LIMIT THEOREMS FOR RANDOM WALK IN A MIXING RANDOM ENVIRONMENT

by

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A dissertation submitted to the faculty of
The University of Utah
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

The University of Utah

August 2012

The University of Utah Graduate School

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ABSTRACT

We consider a random walk on $\mathbb{Z}^{d+1}$ in a cone-mixing space-time random environment. We give a condition for a law of large numbers to hold. Furthermore, assuming an exponentially decreasing spatial-mixing condition, as well as an exponentially decreasing cone-mixing condition, an almost-sure quenched functional central limit theorem is proved by using a martingale approach.

For Eric,

who has been with me every step of the way.

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ACKNOWLEDGEMENTS

I am grateful for the wisdom and patience of my advisor, without whose advice, mentoring, and encouragement during the past years, this research would have been impossible.

I would like to thank my cousin, Kim, who not only encouraged me to start the process of getting my Ph.D., but also convinced me to take probability and statistics classes, which ultimately led to my research area.

Also, thank you to those who have encouraged me to keep going when things got difficult (and provided me with company during regular visits to coffee shops), especially Elisha, Erin, and Hannah.

Thank you to all my math teachers, especially Ms. Biagiotti, who never let me settle for "good enough" back when I thought that math was too easy, and Dr. Stout, who encouraged me as I prepared for graduate school.

I am very grateful for my family and their support throughout the past few years. With my parents', grandparents', and in-laws' encouragement and support, I have been able to keep working. Also, thank you to my brother-in-law, Karl, whose sense of humor has provided several good laughs throughout this process. Through the example and wisdom of my brother, Tim, I have learned to keep going, even through the darkest times when all efforts feel like a chase after the wind.

Above all, I would like to thank my husband, who has patiently been supporting and encouraging me through every step of this process since the day that we met.

CHAPTER 1

INTRODUCTORY MATERIAL

1.1 Introduction

Random walk in random environment (RWRE) is commonly used to model motion in disordered media. Such probabilistic models originated in physical sciences, such as physical chemistry, solid state physics, biophysics, and even oceanography. Diffusion in homogeneous materials can be well modeled by ordinary random walk, but inhomogeneities in the medium are more realistic. However, adding the extra level of randomness makes even simply stated models behave in unexpected manners.

A basic example of this unexpected behavior occurs already in the one-dimensional case. If we consider a nearest-neighbor random walk $X = (X_n)_{n\geq 0}$ in one-dimension with the probability to move right p and the probability to move to the left $q = 1-p$, the average velocity of the walk is $v = \lim_{n\to\infty} \frac{X_n}{n} = p - q$. In the comparable random walk in random environment, given the environment ω, denote the probability to move right $p = p(\omega)$ and the probability to move left $q = q(\omega)$. Letting $\mathbb{E}$ represent expectation under $\mathbb{P}$, the probability governing the environments, one would expect that the velocity is $v = \mathbb{E}[p - q]$. However, this is one of the many instances where intuition misleads us. Solomon showed in [48] how to handle this situation. Denoting $\rho = q/p$, we know by Jensen's inequality that $\mathbb{E}[\rho]^{-1} \leq \mathbb{E}[\rho^{-1}]$. The average velocity of the walk is:

$$v = \lim_{n\to\infty} \frac{X_n}{n} = \begin{cases} \frac{1-\mathbb{E}[\rho]}{1+\mathbb{E}[\rho]} & \text{if } \mathbb{E}[\rho]^{-1} > 1, \\ 0 & \text{if } \mathbb{E}[\rho]^{-1} \leq 1 \leq \mathbb{E}[\rho^{-1}], \\ \frac{\mathbb{E}[\rho^{-1}]-1}{\mathbb{E}[\rho^{-1}]+1} & \text{if } \mathbb{E}[\rho^{-1}] < 1. \end{cases}$$

A consequence of this formula for velocity is quite interesting. Solomon also showed conditions on transience and recurrence for this model. Let P_0 represent the probability that the random walk started at 0, with the environment averaged out. Then:

$$v > 0 \Leftrightarrow \mathbb{E}[\rho] < 1 \Rightarrow \mathbb{E}[\log \rho] < 0 \Rightarrow P_0\left\{ \lim_{n\to\infty} X_n = \infty \right\} = 1,$$

$$v < 0 \Leftrightarrow \mathbb{E}[\rho^{-1}] < 1 \Rightarrow \mathbb{E}[\log \rho] > 0 \Rightarrow P_0\left\{ \lim_{n\to\infty} X_n = -\infty \right\} = 1.$$

Note that it is possible for $\mathbb{E}[\log\rho] \neq 0$ while $\mathbb{E}[\rho]^{-1} \leq 1 \leq \mathbb{E}[\rho^{-1}]$, so the walk can be transient while $v = 0$. This is due to traps that form in the environment. The higher dimensional version of this is directional transience (that is, for some unit vector $\hat{u}$, $X_n \cdot \hat{u} \to \infty$ a.s. while $v = 0$), but the conjecture is that this does not happen (in an i.i.d. environment, for example). Differences such as these between random walk and RWRE, as well as other surprising calculations, have added intrigue to the field.

As a fairly recent field, developed within the last four decades, there are still many open problems and unknown phenomena to be explored. Good overviews of the subject are given by Zeitouni in [57], Bolthausen and Sznitman in [13], and Sznitman in [50, 54].

1.2 Introduction of the Model and Notation

Transition probabilities are denoted by $\pi_{x,y}$ for $x, y \in \mathbb{Z}^{d+1}$, which represent the probability of transitioning from x to y. An environment consists of transition probabilities, $\omega = (\pi_{x,x+z})_{x,z\in\mathbb{Z}^{d+1}} \in \mathcal{P}^{\mathbb{Z}^{d+1}}$ where $\mathcal{P} = \{(p_z)_{z\in\mathbb{Z}^{d+1}} \in [0,1]^{\mathbb{Z}^{d+1}} : \sum_z p_z = 1\}$ and $1 \leq d \in \mathbb{Z}$. Define $\mathbb{Z}_+$ as the set of non-negative integers. We will use $\Omega = \mathcal{P}^{\mathbb{Z}^{d+1}}$ to represent the space of all transition probabilities in dimension $d+1$. The space Ω is Polish and its Borel σ-algebra is the product σ-algebra $\mathfrak{S}$. Ω comes with the shift $\pi_{x,y}(T^z\omega) = \pi_{x+z,y+z}(\omega)$ for all $z \in \mathbb{Z}^{d+1}$. Let ω_x represent the environment at $x \in \mathbb{Z}^{d+1}$, $(\pi_{x,x+z})_z$. For $k \in \mathbb{Z}$, define $\mathfrak{S}_k = \sigma(\omega_x : x \cdot \hat{u} \geq k)$, the σ-algebra generated by the environment in the upper half-space $H_k = \{x : x \cdot \hat{u} \geq k\}$. We are given a T-invariant probability measure $\mathbb{P}$ on $(\Omega, \mathfrak{S})$ with $(\Omega, \mathfrak{S}, (T^z)_{z\in\mathbb{Z}^{d+1}}, \mathbb{P})$ ergodic, i.e., all shift-invariant sets A satisfy $\mathbb{P}(A) \in \{0,1\}$. We will use $\mathbb{E}$ to represent the expectation under $\mathbb{P}$.

We say that an environment is i.i.d. if $\mathbb{P}$ is a product measure with the random probability vectors ω_x i.i.d. For our results, we will not be dealing with i.i.d. environments, but rather environments that differ from being i.i.d. by a small amount. We will use techniques developed for the analysis of RWRE in i.i.d. environments to analyze RWRE in environments that have dependence in both time and space.

Let us now describe the process. The environment ω is chosen according to the distribution $\mathbb{P}$ and then remains fixed. Under P_x^ω, which denotes that the walk starts at the site x under the fixed environment ω, the walk $X = (X_n)_{n\geq 0}$ satisfies the following conditions:

$$P_x^\omega\{X_0 = x\} = 1,$$
$$P_x^\omega\{X_{n+1} = z \,|\, X_n = y\} = \pi_{yz}(\omega),$$

$$P_x^\omega\{X_n \in A\} = P_0^{T_x\omega}\{X_n + x \in A\}.$$

We will refer to P_x^ω as the quenched measure, $P_x = \int P_x^\omega\, \mathbb{P}(d\omega)$ as the joint measure, and its marginal on the sequence space $(\mathbb{Z}^{d+1})^{\mathbb{Z}_+}$ as the averaged measure, also denoted by P_x. We will denote the quenched and averaged expectations by E_x^ω and E_x, respectively. Note that when the environment ω is fixed, the random walk X starting at $x \in \mathbb{Z}^{d+1}$ is a Markov chain, but if the environment is averaged out according to $\mathbb{P}$, the walk is no longer Markovian.

Example 1 *The non-Markovian nature of the averaged random walk in random environment.*

For the one-dimensional case, consider two coins, a red coin and a blue coin. With probability 0.3, the red coin comes up heads and the walker moves to the right one spot. Otherwise, the walker moves to the left one site. In other words, $p_1^{\text{Red}} = 0.3$ and $p_{-1}^{\text{Red}} = 0.7$. With probability 0.6, the blue coin comes up heads and the walker moves to the right two spots. Otherwise, the walker moves to the left one, so $p_2^{\text{Blue}} = 0.6$ and $p_{-1}^{\text{Blue}} = 0.4$. Each site in $\mathbb{Z}$ is assigned either a red or blue coin according to another weighted coin. With probability 0.8, this coin comes up heads, and a red coin is assigned. Otherwise, a blue coin is assigned. Once a coin is assigned to a site, it is permanent, but the red or blue coin is flipped to determine the walker's next move each time he gets to the site. Then, the quenched probability of going from 0 to 1 to 0 to 1 is

$$P_0^\omega\{X_1 = 1, X_2 = 0, X_3 = 1\} = \pi_{01}(\omega)\pi_{10}(\omega)\pi_{01}(\omega) = \pi_{01}(\omega)^2\pi_{10}(\omega).$$

The averaged probability of the same event is

$$\begin{aligned} P_0\{X_1 = 1, X_2 = 0, X_3 = 1\} &= \mathbb{E}\big[P_0^\omega\{X_1 = 1, X_2 = 0, X_3 = 1\}\big] \\ &= \mathbb{E}\big[\pi_{01}(\omega)^2\pi_{10}(\omega)\big] \\ &= 0.8 \cdot 0.3^2 \cdot 0.8 \cdot 0.7 + 0.8 \cdot 0.3^2 \cdot 0.2 \cdot 0.4 \\ &\quad + 0.2 \cdot 0 \cdot 0.8 \cdot 0.7 + 0.2 \cdot 0 \cdot 0.2 \cdot 0.4 \\ &= 0.04608. \end{aligned}$$

Similarly, the probability of going from 0 to 1 to 0 is

$$P_0\{X_1 = 1, X_2 = 0\} = \mathbb{E}\big[\pi_{01}(\omega)\pi_{10}(\omega)\big]$$

$$= 0.8 \cdot 0.3 \cdot 0.8 \cdot 0.7 + 0.8 \cdot 0.3 \cdot 0.2 \cdot 0.4$$
$$= 0.1536.$$

Therefore, the conditional probability of going from 0 to 1, given that the walk has gone from 0 to 1 to 0 is

$$P_0\{X_3 = 1 \,|\, X_1 = 1, X_2 = 0\} = \frac{P_0\{X_1 = 1, X_2 = 0, X_3 = 1\}}{P_0\{X_1 = 1, X_2 = 0\}} = 0.3. \tag{1.1}$$

The probability of going from 0 to 1 is

$$P_0\{X_1 = 1\} = 0.8 \cdot 0.3 = 0.24. \tag{1.2}$$

Since (1.1) and (1.2) are not equal, we see that the averaged process is not Markov.

In this work, we deal with space-time walks. That is, for the standard unit vectors e_i, there exists exactly one $i \in \{1, \ldots, d+1\}$ such that,

$$\mathbb{P}\big\{\pi_{0,z} > 0\big\} > 0 \text{ if and only if } z \cdot e_i = 1. \tag{1.3}$$

We will denote $\hat{u} = e_i$ for this direction, for which we can assume $i = d+1$. This direction represents time, since $P_0\{X_n \cdot \hat{u} = n\} = 1$ for all n. The remaining d dimensions will be referred to as space.

In addition, let the walk be a nearest-neighbor walk on $\mathbb{Z}^d \times \mathbb{Z}$. More precisely,

$$\mathbb{P}\big\{\pi_{0,z} > 0\big\} > 0 \text{ if and only if } z \in \{\hat{u} \pm e_j : 1 \le j \le d\}. \tag{1.4}$$

Our results can be extended to random walk with finite step size; however, we will omit these calculations due to the notation and technical details increasing tremendously.

We also assume that this walk satisfies a uniform ellipticity condition, so the probability that X will go in any spatial direction at any step in time is bounded uniformly away from 0. More precisely, there exists a constant $\kappa > 0$ such that,

$$\mathbb{P}\big\{\omega : \pi_{0,\hat{u}\pm e_j}(\omega) > \frac{\kappa}{2d} \text{ for all } j \text{ such that } e_j \neq \hat{u}\big\} = 1. \tag{1.5}$$

Furthermore, we will assume a cone-mixing condition. Given a set $B \subset \mathbb{Z}^d$, let $\omega_B = \{\omega_x : x \in B\}$ and $\mathfrak{S}_B = \sigma(\omega_B)$. Define $C_x^- = \{y \in \mathbb{Z}^{d+1} : |y - x| \le -(y - x) \cdot \hat{u}\}$, the cone containing all possible nearest-neighbor space-time paths that end at x. The cone-mixing function needed for the Law of Large Numbers (LLN), Φ^-, is defined by

$$\Phi^-(L) = \sup_{\substack{A\in\mathfrak{S}_{C_0^-}\\ B\in\mathfrak{S}_L\\ \mathbb{P}\{A\}\neq 0,\ \mathbb{P}\{B\}\neq 0}} \left|\frac{\mathbb{P}\{A|B\}}{\mathbb{P}\{A\}} - 1\right|.$$

Let $C_x^+ = \{y \in \mathbb{Z}^{d+1} : |y - x| \leq (y - x)\cdot\hat{u}\}$, the cone containing all possible nearest-neighbor space-time paths for X with $X_0 = x$. The function Φ^+ to describe cone-mixing needed for the Central Limit Theorem (CLT) is

$$\Phi^+(L) = \sup_{\substack{A\in\mathfrak{S}_{C_0^+}\\ B\in\mathfrak{S}_{H^c_{-L}}\\ \mathbb{P}\{A\}\neq 0,\ \mathbb{P}\{B\}\neq 0}} \left|\frac{\mathbb{P}\{A|B\}}{\mathbb{P}\{A\}} - 1\right|.$$

Due to the similarity of these functions, many examples result in $\Phi^- = \Phi^+$. For the LLN in Chapter 2, we will assume that $\Phi^-(L)$ is bounded. For the CLT in Chapter 3, we will need the stronger assumption that both $\Phi^-(L)$ and $\Phi^+(L)$ decrease like $e^{-\lambda L}$ for some large constant $\lambda > 0$.

In addition to the cone-mixing functions, we will need to define a spatial mixing function for the CLT. For this, we will first fix $\ell \in \mathbb{Z}$ with $\ell > 0$. Let H represent the lower half-space and define

$$\mathcal{A}_\ell = \{A \subset H : \exists\ z \in A \text{ s.t. } z\cdot\hat{u} = 0 \text{ and } |z| = \ell\}. \tag{1.6}$$

Let F represent the space of local bounded functions f that are measurable on environments in the cone C_0^+. For a fixed set $A \in \mathcal{A}_\ell$, we will let ω and $\tilde{\omega}$ represent environments that differ only at a site $z \in A$ with $z\cdot\hat{u} = 0$ and $|z| = \ell$. Define $\omega_{\not z} = \{\omega_x : x \neq z\}$. Let the spatial mixing function Ψ be the minimal function such that, for $\mathbb{P}$-a.e. ω, and $\mathbb{P}^{\omega_{\not z}}$-a.e. $\tilde{\omega}_z$,

$$\left|\mathbb{E}[f\,|\,\mathfrak{S}_A](\omega) - \mathbb{E}[f\,|\,\mathfrak{S}_A](\tilde{\omega})\right| \leq \Psi(\ell)\|f\|_\infty.$$

for all $A \in \mathcal{A}_\ell$ and $f \in F$. We will need to assume that $\Psi(\ell)$ is exponentially decreasing in ℓ for the CLT to hold.

1.3 Examples

In many mixing situations, the formulas for Φ^- and Ψ can be simplified, bounded, or even calculated exactly. We will not calculate Φ^+ since $\Phi^+ = \Phi^-$ in many circumstances, including the ones considered here. We will consider when environments are i.i.d., in a Gibbs field with the Dobrushin-Shlosman strong decay property, as well as other situations.

Example 2 $\Phi^-(L)$ *in the case of environments that are i.i.d. in time.*

Let $L \geq 1$. Let $A \in \mathfrak{S}_{C_0^-}$ and $B \in \mathfrak{S}_L$ with $\mathbb{P}\{A\} \neq 0$ and $\mathbb{P}\{B\} \neq 0$. Then A and B are independent, so

$$\begin{aligned}\left|\frac{\mathbb{P}\{A|B\}}{\mathbb{P}\{A\}} - 1\right| &= \left|\frac{\mathbb{P}\{A \cap B\}}{\mathbb{P}\{A\}\mathbb{P}\{B\}} - 1\right| \\ &= \left|\frac{\mathbb{P}\{A\}\mathbb{P}\{B\}}{\mathbb{P}\{A\}\mathbb{P}\{B\}} - 1\right| = 0.\end{aligned}$$

Since the choice of A and B was arbitrary from the given σ-algebras, $\Phi^-(L) = 0$ for all $L \geq 1$. Note that the case where environments are i.i.d. in both space and time is included in this example.

Example 3 $\Phi^-(L)$ *when the environment is M-dependent in direction $\hat{u}$.*

By the definition of M-dependence, $\sigma(\omega_x; x \cdot \hat{u} \leq 0)$ and $\sigma(\omega_x; x \cdot \hat{u} \geq M)$ are independent. Let $L \geq M$. Then for $A \in \mathfrak{S}_{C_0^-}$ and $B \in \mathfrak{S}_L$, A and B are independent, and

$$\left|\frac{\mathbb{P}\{A|B\}}{\mathbb{P}\{A\}} - 1\right| = 0,$$

as in the i.i.d. in time case, so $\Phi^-(L) = 0$ for $L \geq M$.

Example 4 $\Psi(\ell)$ *when environments are i.i.d. in both space and time.*

In the i.i.d. case, for any $\ell \geq 1$ and $A \in \mathcal{A}_\ell$, where $\mathcal{A}_\ell$ is as defined in (1.6), we can calculate the following:

$$\begin{aligned}\left|\mathbb{E}[f \mid \mathfrak{S}_A](\omega) - \mathbb{E}[f \mid \mathfrak{S}_A](\tilde{\omega})\right| &= \left|\mathbb{E}[f \mid \mathfrak{S}_{\{0\}}](\omega) - \mathbb{E}[f \mid \mathfrak{S}_{\{0\}}](\tilde{\omega})\right| \\ &= 0,\end{aligned}$$

since $\omega_0 = \tilde{\omega}_0$. As this holds for all $A \in \mathcal{A}_\ell$ and all local bounded functions f that are measurable on environments in C_0^+, we can conclude that $\Psi(\ell) = 0$ for all $\ell \geq 1$.

Example 5 $\Psi(\ell)$ *in the case with environments that are i.i.d. in time, but dependent in space.*

Fix $\ell \geq 1$. Due to independence in time, we only need to consider sets in $\mathcal{B}_\ell \subset \mathcal{A}_\ell$ defined by $\mathcal{B}_\ell = \{B \in \mathcal{A}_\ell : x \cdot \hat{u} = 0 \ \forall \ x \in B\}$. We will also need to consider only a set G of functions $f \in F$ that are measurable with respect to $\mathfrak{S}_{\{0\}}$, as values of the function above level 0 do not depend on $\mathfrak{S}_B$. We can then simplify the requirements for $\Psi(\ell)$ to

$$\big|\mathbb{E}[f \,|\, \mathfrak{S}_B](\omega) - \mathbb{E}[f \,|\, \mathfrak{S}_B](\tilde{\omega})\big| \leq \Psi(\ell)\|f\|_\infty,$$

for all $B \in \mathcal{B}_\ell$ and $f \in G$.

Example 6 $\Psi(\ell)$ *for a linear combination of i.i.d. random variables.*

Let r be a constant with $0 < r < 1$, and let $(X_i)_{i\geq 1}$ be bounded i.i.d. random variables in $\mathbb{R}$ whose pdf is continuous and differentiable. Define $Y_i = \sum_{k=i}^{\infty} r^{k-i+1} X_k$, and let $y \in \mathbb{R}$. Then, $P\{Y_1 \leq y \,|\, Y_2, Y_3, \ldots\} = P\{Y_1 \leq y \,|\, X_2, X_3, \ldots\}$ since we can construct the Y_i's knowing the X_i's and vice versa. Then, for $k \in \mathbb{Z}_+$,

$$P\{Y_1 \leq y \,|\, Y_2, \ldots, Y_{k-1}, \tilde{Y}_k, Y_{k+1}, \ldots\} = P\{Y_1 \leq y \,|\, X_2, \ldots, X_{k-2}, \tilde{X}_{k-1}, \tilde{X}_k, X_{k+1}, \ldots\},$$

where $\tilde{X}_{k-1} = r^{-2}(Y_{k-1} - \tilde{Y}_k)$ and $\tilde{X}_k = r^{-1}(\tilde{Y}_k - Y_{k+1})$. Using this and that $X_1 = r^{-1}(Y_1 - r^2 X_2 - \ldots - r^{k-1}X_{k-1} - r^k X_k - \ldots)$, we get that

$$\begin{aligned}
&\big|P\{Y_1 \leq y \,|\, Y_2, \ldots, Y_{k-1}, Y_k, Y_{k+1}, \ldots\} - P\{Y_1 \leq y \,|\, Y_2, \ldots, Y_{k-1}, \tilde{Y}_k, Y_{k+1}, \ldots\}\big| \\
&= \big|P\big\{X_1 \leq r^{-1}(y - r^2 x_2 - \ldots - r^{k-1}x_{k-1} - r^k x_k - \ldots)\big\} \\
&\quad - P\big\{X_1 \leq r^{-1}(y - r^2 x_2 - \ldots - r^{k-1}\tilde{x}_{k-1} - r^k \tilde{x}_k - \ldots)\big\}\big| \\
&= P\big\{X_1 \in I\big\},
\end{aligned}$$

where I is an interval of length $|r^{k-2}(\tilde{x}_{k-1} - x_{k-1}) - r^{k-1}(\tilde{x}_k - x_k)| \leq Cr^k$ since each X_i is bounded. Since the pdf of the X_i's is bounded, we see that

$$\big|P\{Y_1 \leq y \,|\, Y_2, \ldots, Y_{k-1}, Y_k, Y_{k+1}, \ldots\} - P\{Y_1 \leq y \,|\, Y_2, \ldots, Y_{k-1}, \tilde{Y}_k, Y_{k+1}, \ldots\}\big| \leq Cr^k,$$

so $\Psi(\ell) \leq Cr^\ell$. A similar result also holds in the multidimensional case.

Example 7 *Space-time walk in a Gibbs field with the Dobrushin-Shlosman strong decay property.*

An example of a space-time walk with exponential temporal and spatial mixing is a space-time walk in a Gibbs field.

For a probability measure Q, let Q_V represent the projection of Q onto $(\Omega_V, \mathfrak{S}_V)$. For $\Lambda \subset V$, let $Q_{V,\Lambda}$ be defined as the projection of Q_V onto $(\Omega_\Lambda, \mathfrak{S}_\Lambda)$. Let $\mathrm{dist}(x, V) =$

$\inf\{|x-y| : y \in V\}$ for some choice of norm $|\cdot|$ on $\mathbb{R}^{d+1}$. Define $\partial_r V = \{x \in \mathbb{Z}^{d+1}\backslash V : \mathrm{dist}(x,V) \leq r\}$, the boundary of size r around V. An r-specification ($r \geq 0$) is a system of functions $Q = \{Q_V^\omega(\cdot) : V \subset \mathbb{Z}^{d+1}, |V| < \infty\}$, such that for all $\omega \in \Omega$, Q_V^ω is a probability measure on $(\Omega_V, \mathfrak{S}_V)$, and, for all $A \in \mathfrak{S}_V$, $Q_V^\omega(A)$ is $\mathfrak{S}_{\partial_r V}$-measurable. A specification Q is called self-consistent if, for any finite Λ, V with $\Lambda \subset V \subset \mathbb{Z}^{d+1}$, one has $(Q_V^\omega)_\Lambda^{\tilde{\omega}_V} = Q_\Lambda^{(\omega_{V^c}, \tilde{\omega}_V)}$ for Q_V^ω-a.e. $\tilde{\omega}_V$.

The Dobrushin-Shlosman strong decay property says that for a self-consistent r-specification Q, there exist constants $C, c > 0$ such that for all finite $\Lambda \subset V \subset \mathbb{Z}^{d+1}$, $x \in \partial_r(V)$, and $\omega, \tilde{\omega} \in \Omega$ with $\omega_y = \tilde{\omega}_y$ when $y \neq x$, we have that

$$d_{\mathrm{Var}}(Q_{V,\Lambda}^\omega, Q_{V,\Lambda}^{\tilde{\omega}}) \leq C e^{-c\,\mathrm{dist}(x,\Lambda)}, \tag{1.7}$$

where d_{Var} is the variational distance between two measures.

The condition in (1.7) is satisfied for several classes of Gibbs fields, including in high temperature, large magnetic fields, and one-dimensional and almost one-dimensional interactions. For further discussion of Gibbs fields, see [21].

Example 8 *An example of a Gibbs field with Dobrushin-Shlosman mixing: The Ising Model.*

At each site in $\mathbb{Z}^{d+1}$, assign a 1 or -1 (corresponding to a spin up or down), with probability $1/2$, independently at all sites. We will call this i.i.d. measure P, and the configuration σ.

Given a finite volume $V \subset \mathbb{Z}^{d+1}$, and the configuration outside the volume σ_{V^c}, define the energy $H_V(\sigma_V \,|\, \sigma_{V^c})$ of a configuration inside the volume as the sum of the products of all nearest-neighbor pairs of spins with at least one site in the volume:

$$H_V(\sigma_V \,|\, \sigma_{V^c}) = \sum_{\substack{i,j \in \mathbb{Z}^{d+1} \\ |i-j|=1 \\ i \in V \text{ or } j \in V}} \sigma_i \sigma_j.$$

Fix an inverse temperature, $\beta > 0$. Define a measure $Q_V^{\sigma_{V^c}}$ on the configurations in this volume V, given the configuration outside the volume as

$$dQ_V^{\sigma_{V^c}} = \frac{e^{\beta H_V(\sigma_V \,|\, \sigma_{V^c})}}{Z_V(\beta, \sigma_{V^c})}\, dP,$$

where

$$Z_V(\beta, \sigma_{V^c}) = \sum_{\sigma_V \in \{\pm 1\}^V} e^{\beta H_V(\sigma_V \,|\, \sigma_{V^c})}$$

is a normalization factor.

If β is small enough, there exists a unique probability measure Q on $\{\pm1\}^{\mathbb{Z}^{d+1}}$ such that $Q_V^{\sigma_{V^c}}$ is the regular conditional probability of Q given σ_{V^c}, for all V and Q-a.e. σ_{V^c}. Moreover, Q is shift-invariant and ergodic. See Section 5.2 of [27]. Furthermore, if β is small enough, then (1.7) holds. See [21].

Now, we can choose probability vectors $p, q \in \mathcal{P}$ and let $\omega_x = p$ if $\sigma_x = 1$ and $\omega_x = q$ if $\sigma_x = -1$. $\mathbb{P}$ is then the law of ω.

Example 9 $\Phi^-(L)$ *in the case of a Gibbs field with the Dobrushin-Shlosman strong decay property, as described in Example 7.*

Let $A \in \mathfrak{S}_{C_0^-}$. Then, by Lemma 7 of [41], we have that

$$\begin{aligned}\frac{\mathbb{P}\{A|B\}}{\mathbb{P}\{A\}} &\le \exp\left(C \sum_{\substack{x\in\partial_r(B^c)\\ y\in\partial_r(A^c)}} e^{-c\,\mathrm{dist}(x,y)}\right)\\ &\le 1 + Ce^{-cL},\end{aligned}$$

where C and c are positive constants. The other direction follows similarly, and

$$\left|\frac{\mathbb{P}\{A|B\}}{\mathbb{P}\{A\}} - 1\right| \le Ce^{-cL}.$$

As a result, $\Phi^-(L) \le Ce^{-cL}$.

Example 10 $\Psi(\ell)$ *in the case of a Gibbs field with the Dobrushin-Shlosman strong decay property.*

For a Gibbs r-specification Q, changing the environment at one site has a minimal effect on distant sites. Dobrushin and Shlosman showed in [21] that under (1.7) there exists a unique $\mathbb{P}$ that is consistent with Q. In other words, Q_V^ω is the regular conditional probability of $\mathbb{P}$ given ω_V. From the Dobrushin-Shlosman condition (1.7), it follows that there exist constants $C, c > 0$ such that for all events $E \in \mathfrak{S}_{C_0^+}$, and for all $z \notin C_0^+$ with $\mathrm{dist}(z, C_0^+) > r$, and $\omega, \tilde\omega \in \Omega$ with $\omega_y = \tilde\omega_y$ when $y \neq z$,

$$\left|\mathbb{P}^\omega\{E\} - \mathbb{P}^{\tilde\omega}\{E\}\right| \le Ce^{-c\,\mathrm{dist}(z,C_0^+)}.$$

Fix a set $A \in \mathcal{A}_\ell$. Finding $z \in A$ with $z \cdot \hat{u} = 0$ and $|z| = \ell$ (so that $\mathrm{dist}(z, C_0^+) = \ell$) and by Lemma 7 of [41], we see that

$$\left|\mathbb{E}[f \mid \mathfrak{S}_A](\omega) - \mathbb{E}[f \mid \mathfrak{S}_A](\tilde\omega)\right| \le Ce^{-c\ell}\|f\|_\infty$$

for $\ell > r$. As a result, $\Psi(\ell) \le Ce^{-c\ell}$.

1.4 Previous Results

Many results proving laws of large numbers and invariance principles have already been shown for RWRE. The one-dimensional case was considered in great depth after the aforementioned work of Solomon. These results include showing that the limiting velocity holds not just for i.i.d. environments, but also for ergodic environments; see the work of Alili in [1]. Averaged central limit theorems have been shown in i.i.d. environments when $v > 0$ given $\mathbb{E}[\rho^2] < 1$ by Kesten, Kozlov, and Spitzer in [33]. Furthermore, they showed that if $v = 0$ and $\mathbb{E}[\rho^{-1}] > 0$ under the condition $\mathbb{E}[\log \rho] < 0 < \log \mathbb{E}[\rho]$, then $\mathbb{E}[\rho^s] = 1$ for some $s \in (0, 1)$ and $n^{-s} X_n$ converges in law. Goldsheid showed in [28] sufficient conditions for the environment, under which a quenched Central Limit Theorem holds in a nearest-neighbor one-dimensional case. In [39], Peterson showed that in an i.i.d. environment with a walk with speed $v > 0$, and the averaged law of $n^{-1/s}(X_n - nv)$ converging to a stable law of parameter s for $s \in (1, 2)$, no limit laws are possible. Specifically, there exist sequences depending on the environment such that a quenched CLT holds along a subsequence, but along another subsequence, the limiting distribution is a centered reverse exponential distribution. Recently, Peterson and Samorodnitsky in [40], and independently Dolgopyat and Goldsheid [22], proved that for transient nearest-neighbor one-dimensional RWRE the quenched distribution of hitting times have a stable limit law in the weak sense. For nearest-neighbor RWRE in $\mathbb{Z}$, Enriquez, Sabot, Tournier, and Zindy showed a quenched limit theorem for the hitting time of a level n in [25].

Work in the multidimensional case started more recently. Kalikow has a good discussion of RWRE in multiple dimensions, and addresses transience conditions and zero-one laws in [32]. For static environments, Sznitman and Zerner in [56] showed that a LLN holds for RWRE under conditions discussed by Kalikow, implying directional transience. Sznitman later showed an averaged central limit theorem under Kalikow's condition and considered tail estimates on the probability of slowdowns, giving insight into traps in the medium in [51]. Later, he showed that laws of large numbers and averaged central limit theorems hold in certain ballistic environments, as well as giving an effective criterion where the LLN and CLT hold in [52, 53]. Kipnis and Varadhan showed that an invariance principle holds for additive functionals of reversible Markov chains under certain moment conditions in [34]. Maxwell and Woodroofe in [38] and also Derriennic and Lin in [19] extended this to the non-reversible setting. Using these ideas, quenched invariance principles for the space-time case and the ballistic case were shown by Rassoul-Agha and Seppäläinen in

[42, 44], respectively, when environments were independently assigned. Berger and Zeitouni showed in [9] that every random walk in an i.i.d. environment in dimension $d \geq 2$ that satisfies an averaged invariance principle and an integrability condition for regeneration times also satisfies a quenched invariance principle when the walk has an almost sure positive speed in some direction. Zerner showed that a LLN still holds for i.i.d. random environments when $v = 0$ in a given direction in [58].

Many results have also been shown for dynamic environments. In [10], Boldrighini, Minlos, and Pellegrinotti showed that an almost sure CLT holds for a Markov random environment with $d \geq 2$. Later, in [11], they showed a quenched invariance principle for i.i.d. space-time environments. In [23], Dolgopyat, Keller, and Liverani proved a quenched CLT for random walks with bounded increments, where the evolution of the environment is Markovian with strong spatial and temporal mixing. Dolgopyat and Liverani in [24] showed a quenched CLT for a random walk with environments that satisfy a deterministic and strongly chaotic evolution. In [5], Bandyopadhyay and Zeitouni proved an averaged strong LLN and invariance principle for any dimension, and furthermore showed a quenched invariance principle in high dimensions ($d > 7$) for space-time random walks in Markovian fields (not just Markov in time). Avena, dos Santos, and Völlering recently showed a LLN for a space-time nearest-neighbor $1+1$-dimensional RWRE driven by a symmetric exclusion process in [4]. Andres showed a quenched invariance principle using heat kernel estimates for a dynamic random conductance model in [2].

Invariance principles have also been shown on structures other than $\mathbb{Z}^d$, such as infinite percolation clusters in multiple dimensions. Sidoravicius and Sznitman in [47] showed that an almost sure quenched invariance principle holds for simple symmetric random walk on the infinite Bernoulli percolation clusters on $\mathbb{Z}^d$ with $d \geq 4$. Later, Berger and Biskup in [7], and independently, Mathieu and Piatnitski in [37], extended this result for $d \geq 2$.

In cases of $v = 0$, several results have been found. Lawler proved a quenched invariance principle in the case of a balanced RWRE on $\mathbb{Z}^d$ for a uniformly elliptic environment in [35]. Uniform ellipticity was recently removed by Guo and Zeitouni in [30], then ellipticity was removed altogether by Berger and Deuschel in [8]. Bricmont and Kupiainen in [15] showed that for small random perturbations of a simple random walk, the walk remains diffusive for almost all environments in $\mathbb{Z}^d$ with $d > 2$. The corresponding scaled path space measures converge weakly to Brownian motion. Later, Bolthausen, Sznitman, and Zeitouni proved a law of large numbers and a functional central limit theorem in [14] without the perturbation methods of Bricmont and Kupiainen. Sznitman and Zeitouni showed that an invariance principle, as well as transience, hold for diffusions that are small random perturbations of Brownian motion for $d \geq 3$ in [55].

Limited results have been proved when environments are mixing. Comets and Zeitouni showed in [17] that a LLN holds in environments under a strict cone-mixing condition given either a non-nestling assumption or Kalikow's condition using the regeneration argument of Sznitman and Zerner. Using methods from spectral analysis, Boldrighini, Minlos, and Pellegrinotti showed in [12] that ergodicity conditions hold, and therefore also a LLN, when there is Markov dependence on time. In [31], Joseph and Rassoul-Agha proved that an invariance principle holds for a space-time random walk in $\mathbb{R}^d \times \mathbb{Z}$ with polynomial mixing in space with i.i.d. time components. Bricmont and Kupiainen showed using a renormalization group scheme that an invariance principle holds for environments that are exponentially mixing in both space and time, but still perturbations of random walks, in [16]. Recently, several have shown that laws of large numbers exists under certain cone-mixing conditions. Den Hollander, dos Santos, and Sidoravicius show a LLN for cone-mixing environments, including nonelliptic examples in [18]. Redig and Völlering in [45] consider the case of a Markovian environment under a coupling condition. They prove concentration inequalities for the environment as seen from the particle so a LLN and CLT follow. In [29], Guo proved a conditional LLN for strong-mixing random Gibbsian environments in $\mathbb{Z}^d$ when $d \geq 2$, as well as showed that there is at most one nonzero limiting velocity in higher dimensions $(d \geq 5)$. We use a different technique than Avena, den Hollander, and Redig in [3] to show a LLN, and we will furthermore show that an invariance principle holds under a cone-mixing condition. We do not assume that the environment is Markovian, nor do we need to assume coupling conditions.

Several approaches have been used to show that a LLN holds. Sznitman and Zerner in [56] showed a LLN by using a renewal argument, which was later adapted by several others. In a cone-mixing environment, Comets and Zeitouni in [17] introduced a regeneration-time argument, which was adapted by Avena, den Hollander, and Redig in [3]. The approach we will use here considers the point of view of the particle, and involves showing that there exists an ergodic measure $\mathbb{P}_\infty$ that is invariant for the process $(T^{X_n}\omega)$ such that $\mathbb{P}$ and $\mathbb{P}_\infty$ are mutually absolutely continuous on the upper half-space H_0.

To prove a quenched CLT for RWRE, there are three main approaches. Boldrighini, Minlos, and Pellegrinotti in [11] used Fourier analysis, which requires uniform exponential moments on the steps of the random walk. In other words, $\sup_\omega E_0^\omega(e^{\lambda|X_1|}) < \infty$ for some $\lambda > 0$. A second approach, which was used by Berger and Zeitouni in [9], following the ideas of Bolthausen and Sznitman in [13], uses a concentration inequality to show that the quenched process is close to the averaged process, and then uses the averaged CLT. The third approach, which will be used here, uses general Markov chain arguments by considering the environment as seen by the particle and the Markov chain $(T^{X_n}\omega)$. This method was first used by Kipnis and Varadhan in [34], then was generalized by many, including Maxwell and Woodroofe in [38], Derriennic and Lin in [20], and Rassoul-Agha and Seppäläinen in [43].

CHAPTER 2

THE LAW OF LARGE NUMBERS

2.1 Assumptions and Notation

In this chapter, we will prove a law of large numbers for a cone-mixing nearest-neighbor space-time RWRE on the $d+1$-dimensional integer lattice $\mathbb{Z}^{d+1}$. Our goal is to show that under certain cone-mixing conditions,

$$\mathbb{P}\left\{\lim_{n\to\infty} \frac{X_n}{n} = \mathbb{E}^{\mathbb{P}_\infty}[D]\right\} = 1$$

for some invariant measure $\mathbb{P}_\infty$, where $D(\omega) = E_0^\omega(X_1) = \sum_i \pm e_i \pi_{0,\hat{u}\pm e_i}(\omega)$ is the drift.

Recall the definition of the cone-mixing function from Section 1.2:

$$\Phi^-(L) = \sup_{\substack{A\in\mathfrak{S}_{C_0^-} \\ B\in\mathfrak{S}_L \\ \mathbb{P}\{A\}\neq 0,\ \mathbb{P}\{B\}\neq 0}} \left|\frac{\mathbb{P}\{A|B\}}{\mathbb{P}\{A\}} - 1\right|. \tag{2.1}$$

In addition to the conditions in Section 1.2, we will make the following assumption on the temporal mixing:

Assumption 11 $\Phi^-(L)$ *is bounded for some* $L > 0$.

Define $\mathbb{P}_n\{A\} = P_0\{T^{X_n}\omega \in A\}$, the measure on the environment as seen from the particle at time n, and let $\mathbb{E}_n$ be the expectation under $\mathbb{P}_n$. Denote the quenched expected visits to 0 after n steps of the walk by

$$f_n(\omega) = \sum_{x\in\mathbb{Z}^d} P_x^\omega\{X_n = 0\}.$$

Note that in the nearest-neighbor space-time case, satisfying (1.3) and (1.4), f_n simplifies to

$$f_n(\omega) = \sum_{\substack{x\cdot\hat{u}=-n \\ |x|\le n}} P_x^\omega\{X_n = 0\}. \tag{2.2}$$

Lemma 12 *We have that*

$$\frac{d\mathbb{P}_n}{d\mathbb{P}}(\omega) = f_n(\omega).$$

Proof:

By the definition of $\mathbb{P}_n$, in the space-time case,

$$\begin{aligned}\mathbb{P}_n\{A\} &= \sum_{\substack{x\cdot\hat{u}=n\\|x|\le n}} \int P_0^\omega\{X_n = x\}\mathbb{1}_A(T_x\omega)\,\mathbb{P}(d\omega)\\ &= \sum_{\substack{x\cdot\hat{u}=-n\\|x|\le n}} \int P_x^\omega\{X_n = 0\}\mathbb{1}_A(\omega)\,\mathbb{P}(d\omega),\end{aligned}$$

where we used shift-invariance in the second line. Therefore, $\frac{d\mathbb{P}_n}{d\mathbb{P}}(\omega) = f_n(\omega)$. □

Let $\tilde{\mathbb{P}}_n = n^{-1}\sum_{k=0}^{n-1}\mathbb{P}_k$, the Cesàro mean. Since our space Ω is compact, a subsequence $\{\tilde{\mathbb{P}}_{n_j}\}$ of $\tilde{\mathbb{P}}_n$ will converge to some $\mathbb{P}_\infty$.

Proposition 13 *The Markov process with initial distribution* $\mathbb{P}_\infty$ *and transition* $\bar{\pi}(\omega, T^z\omega) = \pi_{0z}(\omega)$ *is stationary.*

The process $(T^{X_n}\omega)_{n\ge0}$ is Markov with transition $\bar{\pi}$, so Proposition 13 will imply that $\mathbb{P}_\infty$ is invariant for the process $(T^{X_n}\omega)_{n\ge0}$.

Proof:

By the definition of $\mathbb{P}_n$ and Lemma 12,

$$\begin{aligned}\mathbb{P}_n\{A\} &= \int P_0^\omega\{T^{X_1}\omega \in A\}\,\mathbb{P}_n(d\omega)\\ &= \int \sum_z \mathbb{1}\{T^z\omega\in A\}\pi_{0z}(\omega)\sum_{x\in\mathbb{Z}^d} P_x^\omega\{X_n = 0\}\,\mathbb{P}(d\omega)\\ &= \int_A \sum_{x\in\mathbb{Z}^d}\sum_z \pi_{z0}(\omega)P_x^\omega\{X_n = z\}\,\mathbb{P}(d\omega)\\ &= \int_A d\mathbb{P}_{n+1} = \mathbb{P}_{n+1}\{A\},\end{aligned}$$

so the statement is proved. □

2.2 Consequences of the Cone-Mixing Condition

In this section, we will explore the implications of the cone-mixing condition.

Proposition 14 *Assume* $0 < n$ *and* $x\cdot\hat{u} = -n$. *Then the following statements hold* $\mathbb{P}$*-a.s.:*

(a) Let $g \ge 0$ *be* $\mathfrak{S}_{C_x^-}$*-measurable. Then,*

$$\big|\mathbb{E}[g] - \mathbb{E}\big[g\,|\,\mathfrak{S}_0\big]\big| \le \Phi^-(n)\mathbb{E}[g].$$

(b) Let $f \geq 0$ be a $\mathfrak{S}_0$-measurable function and $g \geq 0$ be $\mathfrak{S}_{C_x^-}$-measurable. Then

$$\big|\mathbb{E}[fg] - \mathbb{E}[f]\,\mathbb{E}[g]\big| \leq \Phi^-(n)\mathbb{E}[f]\,\mathbb{E}[g].$$

(c) Let $g \geq 0$ be $\mathfrak{S}_L$-measurable for some $L > 0$. Then, for all n,

$$\big|\mathbb{E}_n[g] - \mathbb{E}[g]\big| \leq \Phi^-(L)\mathbb{E}[g].$$

Proof:

To show statement (a), let $A \in \mathfrak{S}_{C_x^-}$ and $g = \mathbb{1}_A$.

$$\begin{aligned}
\mathbb{E}\big[\mathbb{1}_A \,|\, \mathfrak{S}_0\big] &= \mathbb{P}\{A \,|\, \mathfrak{S}_0\} \\
&\leq \big(1 + \Phi^-(n)\big)\mathbb{P}\{A\} \\
&= \big(1 + \Phi^-(n)\big)\mathbb{E}[\mathbb{1}_A].
\end{aligned}$$

We can extend this to all $\mathfrak{S}_{C_x^-}$-measurable simple functions g by using linear combinations of indicator functions, and can then further extend this to all continuous $\mathfrak{S}_{C_x^-}$-measurable functions by approximating them with simple functions. Similarly, the lower bound also holds.

To prove statement (b), we will start by multiplying the lower bound from statement (a) by f. Using that $f \geq 0$, we get that

$$\big(1 - \Phi^-(n)\big)\, f \cdot \mathbb{E}[g] \leq f \cdot \mathbb{E}\big[g \,|\, \mathfrak{S}_0\big].$$

Taking the expectation under $\mathbb{P}$ on both sides, we have that

$$\begin{aligned}
\big(1 - \Phi^-(n)\big)\, \mathbb{E}[f]\,\mathbb{E}[g] &\leq \mathbb{E}\big[f\mathbb{E}\big[g \,|\, \mathfrak{S}_0\big]\big] \\
&= \mathbb{E}[fg],
\end{aligned}$$

by using that f is $\mathfrak{S}_0$-measurable and the definition of conditional expectation. Likewise, the upper bound holds.

For the proof of statement (c), use the definition of $\mathbb{P}_n$, that $P_x^\omega\{X_n = 0\}$ is $\mathfrak{S}_{C_0^-}$-measurable, and the result of statement (b) to write

$$\begin{aligned}
\mathbb{E}_n[g] &= \sum_{x\cdot\hat{u}=-n} \int P_x^\omega\{X_n = 0\} g(\omega)\, \mathbb{P}(d\omega) \\
&\leq \big(1 + \Phi^-(L)\big) \sum_{x\cdot\hat{u}=-n} \mathbb{E}\big[P_x^\omega\{X_n = 0\}\big]\mathbb{E}[g]
\end{aligned}$$

$$= \big(1 + \Phi^-(L)\big)\mathbb{E}[f_n]\mathbb{E}[g]$$
$$= \big(1 + \Phi^-(L)\big)\mathbb{E}[g],$$

where the last line is by Lemma 12.

In a similar fashion, we get the lower bound. □

Note that, after taking a Cesàro mean, taking $n \to \infty$ in part (c) of the above Proposition, we also get that $|\mathbb{E}_\infty[g] - \mathbb{E}[g]| \le \Phi^-(L)\mathbb{E}[g]$ for bounded continuous $g \ge 0$.

2.3 The Law of Large Numbers

In this section, a law of large numbers for the random walk described in Section 1.2 will be shown by first showing that $\mathbb{P}$ and $\mathbb{P}_\infty$ are mutually absolutely continuous, then that the process $(T^{X_n}\omega)_{n\ge 0}$ with initial distribution $\mathbb{P}_\infty$ is ergodic. This will lead to the law of large numbers in Theorem 19. Uniqueness of $\mathbb{P}_\infty$ will be shown along the way.

Theorem 15 *Suppose that $\mathbb{P}$ satisfies Assumption 11,* (1.4)*, and* (1.5)*. Then $\mathbb{P}_\infty$ is absolutely continuous relative to $\mathbb{P}$ on every half-space H_k with $k \le 0$.*

Proof:
Let $k \in \mathbb{Z}$ and $m, n \in \mathbb{Z}_+$ with $k \le 0$ fixed. Let $-k < m < n$ and define

$$g_n(\omega) = \mathbb{E}\big[f_n \,\big|\, \mathfrak{S}_k\big](\omega) = \frac{d\mathbb{P}_n\,|_{\mathfrak{S}_k}}{d\mathbb{P}\,|_{\mathfrak{S}_k}}(\omega).$$

By considering the possible positions of the walk after m steps, we see that

$$g_n = \mathbb{E}\Big[\sum_{\substack{x\cdot\hat{u}=-n\\|x|\le n}} P_x^\omega\{X_n = 0\}\,\Big|\,\mathfrak{S}_k\Big]$$
$$= \sum_{\substack{y\cdot\hat{u}=-m\\|y|\le m}} \mathbb{E}\Big[\sum_{\substack{x\cdot\hat{u}=-n\\|x|\le n}} P_x^\omega\{X_{n-m} = y\}P_y^\omega\{X_m = 0\}\,\Big|\,\mathfrak{S}_k\Big]$$
$$\le \sum_{\substack{y\cdot\hat{u}=-m\\|y|\le m}} \mathbb{E}\Big[\sum_{\substack{x\cdot\hat{u}=-n\\|x|\le n}} P_x^\omega\{X_{n-m} = y\}\,\Big|\,\mathfrak{S}_k\Big],$$

where the last inequality comes by noting that $P_y^\omega\{X_m = 0\} \le 1$. Then, by Proposition 14, the definition of f, and Lemma 12, the above is bounded $\mathbb{P}$-a.s. by

$$g_n \le \sum_{\substack{y\cdot\hat{u}=-m\\|y|\le m}} \mathbb{E}\Big[\sum_{\substack{x\cdot\hat{u}=-n\\|x|\le n}} P_x^\omega\{X_{n-m} = y\}\Big]\big(1 + \Phi^-(m+k)\big)$$

$$
\begin{aligned}
&= \sum_{\substack{y\cdot\hat{u}=-m\\|y|\le m}} \mathbb{E}[f_{n-m}]\big(1+\Phi^-(m+k)\big)\\
&= \sum_{\substack{y\cdot\hat{u}=-m\\|y|\le m}} \big(1+\Phi^-(m+k)\big)\\
&\le Cm^d\big(1+\Phi^-(m+k)\big),
\end{aligned}
$$

where $C = C(d)$ is constant. This quantity is uniformly bounded if $\Phi^-(m+k)$ is bounded. Note that when n is large, the choice of $m > |k|$ is arbitrary, so this holds when $\Phi^-(L)$ is bounded for some $L > 0$, which holds by Assumption 11.

We now see that if $\hat{\mathbb{P}}_n = \frac{1}{n}\sum_{\ell=m+1}^{n} \mathbb{P}_\ell$, then

$$\frac{d\hat{\mathbb{P}}_n\,|_{\mathfrak{S}_k}}{d\mathbb{P}\,|_{\mathfrak{S}_k}} \le Cm^d\big(1+\Phi^-(m+k)\big)$$

for all $n > m$. Since $\mathbb{P}_\ell \ll \mathbb{P}$ for any fixed ℓ, we observe that

$$\frac{d\mathbb{P}_\infty\,|_{\mathfrak{S}_k}}{d\mathbb{P}\,|_{\mathfrak{S}_k}} \le Cm^d\big(1+\Phi^-(m+k)\big).$$

Consequently, $\mathbb{P}_\infty\,|_{\mathfrak{S}_k} \ll \mathbb{P}\,|_{\mathfrak{S}_k}$. □

Theorem 16 *Suppose that the conditions for Theorem 15 are met. Then $\mathbb{P}$ and $\mathbb{P}_\infty$ are in fact mutually absolutely continuous on every half-space H_k with $k \le 0$.*

Proof:
Fix $k \le 0$, and define

$$G_k = \frac{d\mathbb{P}_\infty|_{\mathfrak{S}_k}}{d\mathbb{P}|_{\mathfrak{S}_k}}.$$

Then

$$
\begin{aligned}
0 &= \int_{\{G_k=0\}} G_k\,d\mathbb{P} = \int \mathbb{1}_{\{G_k=0\}}\,d\mathbb{P}_\infty = \int \sum_z \pi_{0z}\mathbb{1}_{\{G_k=0\}}\circ T^z\,d\mathbb{P}_\infty\\
&= \int \sum_{z\cdot\hat{u}\ge 0} \pi_{0z}\mathbb{1}_{\{G_k=0\}}\circ T^z G_k\,d\mathbb{P} = \int_{\{G_k=0\}} \sum_{z\cdot\hat{u}\ge 0} \pi_{-z,0}G_k\circ T^{-z}\,d\mathbb{P},
\end{aligned}
$$

since if $z\cdot\hat{u} \ge 0$, then $G_k\circ T^z$ is $\mathfrak{S}_k$-measurable. The above implies that $\mathbb{P}$-a.s., $z = \hat{u} \pm e_i$ we have that

$$\{G_k = 0\} \subset T^z\{G_k = 0\},$$

and since T is $\mathbb{P}$-preserving, we have that

$$\{G_k = 0\} = T^z\{G_k = 0\} \quad \mathbb{P}\text{-a.s.}$$

We conclude that $\{G_k = 0\}$ is $\mathbb{P}$-a.s. shift invariant since $\{T^z\,,\, z = \hat{u} \pm e_i, i = 1, \ldots, d\}$ generates $(T^x)_{x \in \mathbb{Z}^d}$. Since $\mathbb{P}$ is ergodic, $\mathbb{P}\{G_k = 0\}$ must be either 0 or 1. Since $\mathbb{E}[G_k] = 1$, $P\{G_k > 0\} = 1$, so $\mathbb{P}$ and $\mathbb{P}_\infty$ are mutually absolutely continuous on every half-space H_k with $k \leq 0$. $\square$

Proposition 17 *The Markov process $(T^{X_n}\omega)_{n \geq 0}$ with initial distribution $\mathbb{P}_\infty$ is ergodic.*

Proof:

Let f be a bounded local function on Ω that is $\mathfrak{S}_K$-measurable for some $K \leq 0$. By Birkhoff's ergodic theorem and bounded convergence, we know that the limit

$$g(\omega) = \lim_{n \to \infty} n^{-1} \sum_{m=1}^{n} E_0^\omega\big(f(T^{X_m}\omega)\big) \tag{2.3}$$

exists $\mathbb{P}_\infty$-a.s. Since g is harmonic (that is, $E_0^\omega(g(T^{X_1}\omega)) = g(\omega)$) and $\mathbb{P}_\infty$ is invariant by Proposition 13, we know that

$$\begin{aligned}\sum_z \int \pi_{0z}(g - g \circ T^z)^2 \, d\mathbb{P}_\infty &= \int g^2 \, d\mathbb{P}_\infty - 2 \int g \sum_z \pi_{0z} g \circ T^z \, d\mathbb{P}_\infty \\ &\quad + \int \sum_z \pi_{0z} (g \circ T^z)^2 \, d\mathbb{P}_\infty \\ &= 0.\end{aligned}$$

By noticing that π_{0z} is $\mathfrak{S}_0$-measurable, g is $\mathfrak{S}_k$-measurable, and the ellipticity condition (1.5), we can conclude that

$$g = g \circ T^z \quad \mathbb{P}\text{-a.s.}$$

for $z = \hat{u} \pm e_i$.

Ergodicity of $\mathbb{P}$ shows that g is constant $\mathbb{P}$-a.s., and now also $\mathbb{P}_\infty$-a.s. Then, $g = \mathbb{E}_\infty[f]$. By L^1 approximation, we have the same result for $g \in L^1(\mathbb{P}_\infty)$, specifically, that

$$\mathbb{P}_\infty\bigg\{ \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} E_0^\omega\big(f(T^{X_m}\omega)\big) = \mathbb{E}_\infty[f] \bigg\} = 1. \tag{2.4}$$

Now, the ergodicity of $\mathbb{P}_\infty$ follows from Section IV.2 of [46]. $\square$

Proposition 18 $\mathbb{P}_\infty$ *is unique: if $\tilde{\mathbb{P}}_n\{A\} = n^{-1} \sum_{m=1}^{n} P_0\{T^{X_m}\omega \in A\}$, then $\tilde{\mathbb{P}}_n$ converges weakly to $\mathbb{P}_\infty$.*

Proof:

Let f be a bounded local $\mathfrak{S}_K$-measurable function for some $K \leq 0$. Then, due to (2.4), we know that $\mathbb{P}_\infty$-a.s.

$$E_0^\omega\left(\lim_{n\to\infty} n^{-1}\sum_{m=1}^{n} f\big(T^{X_m}\omega\big)\right) = \mathbb{E}_\infty\big[f\big].$$

The above is $\mathfrak{S}_K$-measurable, so the same equation holds $\mathbb{P}$-a.s. By integrating over ω, we see that

$$\mathbb{E}_\infty[f] = \lim_{n\to\infty} n^{-1}\sum_{m=1}^{n} E_0\big[f\big(T^{X_m}\omega\big)\big] = \lim_{n\to\infty} \tilde{\mathbb{E}}_n\big[f\big],$$

where $\tilde{\mathbb{E}}_n$ represents the expectation under $\tilde{\mathbb{P}}_n$. Then, $\mathbb{P}_\infty$ is uniquely defined as the weak limit of $\tilde{\mathbb{P}}_n$. $\square$

Theorem 19 *Suppose that $\mathbb{P}$ satisfies Assumption 11,* (1.4)*, and* (1.5)*. Then a law of large numbers holds:*

$$P_0\left\{\lim_{n\to\infty}\frac{X_n}{n} = \mathbb{E}^{\mathbb{P}_\infty}\big[D\big]\right\} = 1.$$

Proof:

Letting f from the proof of Proposition 17 be the drift D, we have for $\mathbb{P}_\infty$-a.e. ω,

$$P_0^\omega\left\{\lim_{n\to\infty} n^{-1}\sum_{m=1}^{n} D\big(T^{X_m}\omega\big) = \mathbb{E}^{\mathbb{P}_\infty}\big[D\big]\right\} = 1.$$

Since the event in question is $\mathfrak{S}_0$-measurable, this is also true $\mathbb{P}$-a.s., and we have that

$$P_0\left\{\lim_{n\to\infty} n^{-1}\sum_{m=1}^{n} D\big(T^{X_m}\omega\big) = \mathbb{E}^{\mathbb{P}_\infty}\big[D\big]\right\} = 1. \tag{2.5}$$

Let $M_n = X_n - X_0 - \sum_{m=0}^{n-1} D(T^{X_m}\omega)$, which is a martingale with bounded increments under P_0^ω. Then, for $\gamma > 0$,

$$\begin{aligned}
P_0^\omega\left\{\frac{M_n}{n} \geq n^{-1/4}\right\} &\leq e^{-\gamma n^{3/4}} E_0^\omega\big(e^{\gamma M_n}\big)\\
&= e^{-\gamma n^{3/4}} E_0^\omega\big(e^{\gamma M_{n-1}} E_{X_{n-1}}^\omega(e^{\gamma M_1})\big)\\
&\leq e^{-\gamma n^{3/4}}\big(1+\mathcal{O}(\gamma^2)\big) E_0^\omega\big(e^{\gamma M_{n-1}}\big)\\
&\leq \cdots \leq e^{-\gamma n^{3/4}}\big(1+\mathcal{O}(\gamma^2)\big)^n\\
&= e^{-\gamma n^{3/4}+\mathcal{O}(n\gamma^2)},
\end{aligned}$$

where the second inequality uses that M_n has bounded increments. Taking $\gamma = n^{-1/2}$ and using the Borel-Cantelli lemma, we get that

$$P_0^\omega\left\{ \lim_{n\to\infty} \frac{M_n}{n} = 0 \right\} = 1.$$

Combining the above with (2.5), we obtain the desired result. □

CHAPTER 3

ALMOST SURE CENTRAL LIMIT THEOREM

3.1 Introduction

Recall the definition of $\Phi^+(L)$ from Section 1.2:

$$\Phi^+(L) = \sup_{\substack{A \in \mathfrak{S}_{C_0^+} \\ B \in \mathfrak{S}_{H_{-L}^c} \\ \mathbb{P}\{A\} \neq 0,\ \mathbb{P}\{B\} \neq 0}} \left| \frac{\mathbb{P}\{A|B\}}{\mathbb{P}\{A\}} - 1 \right|. \tag{3.1}$$

We will show that a functional central limit theorem holds for $\mathbb{P}$-a.e. ω when $\Phi^+(L)$ decays exponentially in L. In order for a CLT to hold, we must also assume a condition on the spatial mixing. If no spatial mixing exists, consider the following counterexample to the CLT:

Example 20 *Counterexample to a.s. CLT without spatial mixing.*

Assign environments $(\omega_{n,0})_{n\geq 0}$ i.i.d. in time, and set $\omega_{n,x} = \omega_{n,0}$ for all $x \in \mathbb{Z}^d$. Let $D(\omega) = E_0^\omega[X_1]$. Then,

$$X_n = \sum_{k=0}^{n-1} D(T^{X_k}\omega) + M_n = \sum_{k=0}^{n-1} D(\omega_{k,0}) + M_n,$$

where M_n is a martingale. Then, $E_0^\omega[X_n] = \sum_{k=0}^{n-1} D(\omega_{k,0})$ and $X_n - E_0^\omega[X_n] = M_n$. Since M_n meets the conditions for the martingale invariance principle, for $\mathbb{P}$-a.e. ω, the law of

$$\frac{X_{[nt]} - E_0^\omega[X_{[nt]}]}{\sqrt{n}}, \quad t \geq 0$$

converges weakly to a Brownian motion under P_0^ω with a covariance matrix independent of ω. However, since there is enough temporal mixing, for $v = E_0[X_1]$, $E_0^\omega(X_n) - nv = \sum_{k=0}^{n-1} \big(D(\omega_{k,0}) - \mathbb{E}[D]\big)$ satisfies its own invariance principle, so the laws of $(X_n - nv)/\sqrt{n}$ are not tight.

For $A \in \mathbb{Z}^d$, recall that $\mathfrak{S}_A = \sigma(\omega_A)$. Let H be the lower half-space. Fix $\ell \in \mathbb{Z}_+$, and define $\mathcal{A}_\ell = \{A \subset H : \exists\ z \in A \text{ s.t. } z \cdot \hat{u} = 0 \text{ and } |z| = \ell\}$. Let F be the space of local bounded functions f that are measurable on environments in the cone C_0^+. Recall that the spatial mixing function defined in Section 1.2, Ψ, is the minimal function such that for all $f \in F$

$$\left|\mathbb{E}[f \,|\, \mathfrak{S}_A](\omega) - \mathbb{E}[f \,|\, \mathfrak{S}_A](\tilde{\omega})\right| \leq \Psi(\ell)\|f\|_\infty, \tag{3.2}$$

for $\mathbb{P}$-a.e. ω and $\tilde{\omega}$ that differ only at a site on level 0 that is ℓ units from the origin. If $\Psi(\ell)$ is small, changing the environment at a site ℓ units away from C_0^+ does not have much of an effect on averaging functions within C_0^+. Due to shift-invariance, note that the choice of using the cone based at 0 was arbitrary.

Let us now consider Ψ in the case of the counterexample discussed in Example 20.

Example 21 $\Psi(\ell)$ *for the counterexample to the CLT described in Example 20.*

Since components are i.i.d. in time, but dependent in space, this is a special case of Example 5. For fixed $\ell \geq 1$, by the formula derived in Example 5,

$$\left|\mathbb{E}[f \,|\, \mathfrak{S}_B](\omega) - \mathbb{E}[f \,|\, \mathfrak{S}_B](\tilde{\omega})\right| \leq \Psi(\ell)\|f\|_\infty.$$

Take $B = \{z\}$ with $z \neq 0$, $\omega_0 \neq \tilde{\omega_0}$, and $f = \mathbb{1}_{\{\omega_0\}}$. Then, $\left|\mathbb{E}[f \,|\, \mathfrak{S}_B](\omega) - \mathbb{E}[f \,|\, \mathfrak{S}_B](\tilde{\omega})\right| = 1$, so $\Psi(\ell) \geq 1$ for all $\ell \geq 1$.

The required conditions on Ψ for a CLT to hold will need to address the difference between the i.i.d. case in Example 4 and the counterexample in Example 21, and should include the linear combination of i.i.d. random variables and Gibbs field considered in Examples 6 and 10, respectively. For the proof of the central limit theorem, we will make the following assumptions on our mixing functions:

Assumption 22 *The temporal mixing functions* Φ^+ *and* Φ^-*, as defined in* (3.1) *and* (2.1)*, respectively, satisfy* $\max\{\Phi^+(L), \Phi^-(L)\} \leq Ce^{-\lambda L}$ *for some* $C > 0$ *constant and* $\lambda > 0$*.*

Assumption 23 *The spatial mixing function* Ψ*, as defined in* (3.2)*, satisfies* $\Psi(\ell) \leq Ce^{-\lambda \ell}$*, where* C *is constant and* $\lambda > 0$*.*

The restrictions on Φ^- and Ψ in Assumptions 22 and 23 are not optimal, and could be improved to polynomial mixing by using more precise bounds throughout these calculations. Also, these techniques can be used to extend the result to walks that backtrack in time.

Given these mixing assumptions, our results will hold for i.i.d. environments by Example 4 as well as for the linear combination of i.i.d. random variables, the Ising Model, and the Gibbs field discussed in Examples 6, 8, and 10, respectively. However, the conditions are not met for the circumstances discussed in Examples 20 and 21.

Let $D_{\mathbb{R}^d}[0,\infty)$ represent the Skorohod space of $\mathbb{R}^d$-valued càdlàg paths. Define the process B_n as

$$B_n(t) = \frac{X_{[nt]} - [nt]v}{\sqrt{n}}$$

and let $Q_n^\omega = P_0^\omega\{B_n \in \cdot\}$ denote the quenched distribution of the process B_n on $D_{\mathbb{R}^d}[0,\infty)$. We will show that for $\mathbb{P}$-a.e. ω the distributions Q_n^ω converge weakly, as $n \to \infty$, to the law of Brownian motion.

Define

$$\tilde{B}_n(t) = \frac{X_{[nt]} - E_0^\omega(X_{[nt]})}{\sqrt{n}}$$

and denote the law of $\tilde{B}_n$ under P_0^ω by $\tilde{Q}_n^\omega$.

Theorem 24 *Assume that the environment measure $\mathbb{P}$ is shift-invariant and satisfies the mixing assumptions 22 and 23. Then for $\mathbb{P}$-a.e. ω, the distributions Q_n^ω converge weakly on $D_{\mathbb{R}^d}[0,\infty)$ to the distribution of Brownian motion with a symmetric non-negative definite diffusion matrix $\mathfrak{D}$, which is independent of ω. Furthermore, $n^{-1/2}\max_{k\le n}|E_0^\omega(X_k) - kv|$ converges to 0 $\mathbb{P}$-a.s. and the same invariance principle holds for the distribution of $\tilde{B}_n$ induced by P_0^ω for $\mathbb{P}$-a.e. ω.*

The proof of this theorem is the ultimate goal of this chapter, and will be shown in Section 3.6.

Define the push-forward of a bounded measurable function h on Ω as $\Pi h(\omega) = \sum_{|z|=1}\pi_{0z}(\omega)h(T^z\omega)$. Let the drift, D, be defined by

$$D(\omega) = E_0^\omega(X_1) = \sum_z z\pi_{0z}(\omega).$$

Define $g = D - v$, where $v = \mathbb{E}_\infty[D]$, as before.

The operator $\Pi - I$ defines the generator of the Markov chain of the environment from the point of view of the particle. The process has transitions

$$\bar{\pi}(\omega, A) = P_0^\omega\{T^{X_1}\omega \in A\}.$$

We say that the measure $\mathbb{P}_\infty$ is stationary for the process $(T^{X_0}\omega, T^{X_1}\omega,\ldots)$ if $(T^{X_0}\omega, T^{X_1}\omega,\ldots)$ and its shift $(T^{X_1}\omega, T^{X_2}\omega,\ldots)$ are equal in $\mathbb{P}_\infty$-distribution. Furthermore, $\mathbb{P}_\infty$ is ergodic for this process if the path measure with initial distribution $\mathbb{P}_\infty$ and transitions $\bar{\pi}$ is ergodic for the above shift.

Theorem 25 *Let $\mathbb{P}_\infty$ be stationary ergodic for the Markov chain with generator $\Pi - I$. Assume $\int E_0^\omega(|X_1|^2)\mathbb{P}_\infty(d\omega) < \infty$. Also, assume that there exists $\alpha \in (0, 1/2)$ such that*

$$\mathbb{E}_\infty\big[\big|E_0^\omega(X_n) - \mathbb{E}_\infty[E_0^\omega(X_n)]\big|^2\big] = \Big\|\sum_{k=0}^{n-1}\Pi^k g\Big\|_2^2 = \mathcal{O}(n^{2\alpha}). \tag{3.3}$$

Then, $n^{-1/2}\max_{k\le n}|E_0^\omega(X_k) - kv|$ converges to 0 $\mathbb{P}_\infty$-a.s., and for $\mathbb{P}_\infty$-a.e. ω, the laws of B_n and $\tilde{B}_n$ under P_0^ω converge weakly to the same Brownian motion with a nonrandom covariance matrix.

This is Theorem 2 of [42], which uses the strategy of Derriennic and Lin in [20] to further extend the result of Maxwell and Woodroofe in [38].

3.2 Step 1: From $\mathbb{P}_\infty$ Back to $\mathbb{P}$

In this section the problem of showing an invariance principle is reduced from showing that $\mathbb{E}_\infty[|E_0^\omega(X_n) - \mathbb{E}_\infty[E_0^\omega(X_n)]|^2] = \mathcal{O}(n^{2\alpha})$ for some $\alpha < 1/2$ to showing that $\mathbb{E}[|E_0^\omega(X_n) - \mathbb{E}[E_0^\omega(X_n)]|^2] = \mathcal{O}(n^{2\bar{\alpha}})$ for some $\bar{\alpha} < 1/2$.

Lemma 26 *Let $k \ge 0$. Then $\mathbb{P}$-a.s.,*

$$\left|1 - \frac{d\mathbb{P}_\infty}{d\mathbb{P}}\Big|_{\mathfrak{S}_k}\right| \le \Phi^-(k).$$

Proof:

By Lemma 12 and Proposition 14, for $f_n(\omega)$ as defined in (2.2),

$$\begin{aligned}\left|1 - \frac{d\mathbb{P}_n}{d\mathbb{P}}\Big|_{\mathfrak{S}_k}\right| = \big|\mathbb{E}[f_n] - \mathbb{E}[f_n\,|\,\mathfrak{S}_k]\big| &\le \mathbb{E}[f_n]\Phi^-(k)\\ &= \Phi^-(k),\end{aligned}$$

since $\mathbb{E}[f_n] = 1$ by Lemma 12.

Taking Cesàro means and $n \to \infty$ on the left-hand side, we get the desired result. □

Proposition 27 *Assume that there exists an $\bar{\alpha} < 1/2$ such that*

$$\mathbb{E}\big[\big|E_0^\omega(X_n) - \mathbb{E}\big[E_0^\omega(X_n)\big]\big|^2\big] = \mathcal{O}(n^{2\bar{\alpha}}). \tag{3.4}$$

Then condition (3.3) is satisfied for some $\alpha < 1/2$.

Proof:

Choose $k = n^\delta$ for some $0 < \delta < 1$. Then, by the Cauchy-Schwartz inequality, we can bound the equation in line (3.3) by

$$\begin{aligned}
&\mathbb{E}_\infty\big[\big|E_0^\omega(X_n) - \mathbb{E}_\infty\big[E_0^\omega(X_n)\big]\big|^2\big] \\
&\leq 2\mathbb{E}_\infty\big[\big|E_0^\omega(X_n) - E_0(X_n)\big|^2\big] + 2\big|\mathbb{E}_\infty\big[E_0^\omega(X_n)\big] - E_0(X_n)\big|^2 \\
&\leq 4\mathbb{E}_\infty\big[\big|E_0^\omega\big(X_n - \mathbb{E}\big[E_0^\omega(X_n)\big]\big)\big|^2\big].
\end{aligned}$$

By restarting the walk at level k, using Jensen's Inequality, and that $P_0^\omega\{X_k = x\} \leq 1$ for all x, we can further bound line (3.3) by

$$\begin{aligned}
&\mathbb{E}_\infty\big[\big|E_0^\omega(X_n) - \mathbb{E}_\infty\big[E_0^\omega(X_n)\big]\big|^2\big] \\
&\leq 4\mathbb{E}_\infty\Bigg[\Bigg|\sum_{\substack{x\cdot\hat{u}=k \\ |x|\leq k}} P_0^\omega\{X_k = x\}E_x^\omega\big(X_{n-k} - \mathbb{E}\big[E_0^\omega(X_n)\big]\big)\Bigg|^2\Bigg] \\
&\leq 4\mathbb{E}_\infty\Bigg[\sum_{\substack{x\cdot\hat{u}=k \\ |x|\leq k}} P_0^\omega\{X_k = x\}\big|E_x^\omega\big(X_{n-k} - \mathbb{E}\big[E_0^\omega(X_n)\big]\big)\big|^2\Bigg] \\
&\leq 4\sum_{\substack{x\cdot\hat{u}=k \\ |x|\leq k}} \mathbb{E}_\infty\big[\big|E_x^\omega(X_{n-k}) - \mathbb{E}\big[E_0^\omega(X_n)\big]\big|^2\big].
\end{aligned}$$

Using Jensen's inequality again, we see that

$$\begin{aligned}
&\mathbb{E}_\infty\big[\big|E_0^\omega(X_n) - \mathbb{E}_\infty\big[E_0^\omega(X_n)\big]\big|^2\big] \\
&\leq 8\sum_{\substack{x\cdot\hat{u}=k \\ |x|\leq k}} \mathbb{E}_\infty\big[\big|E_x^\omega\big(X_{n-k}\big) - \mathbb{E}\big[E_x^\omega(X_{n-k})\big]\big|^2\big] + 8\sum_{\substack{x\cdot\hat{u}=k \\ |x|\leq k}} \big|\mathbb{E}\big[E_x^\omega(X_{n-k})\big] - \mathbb{E}\big[E_0^\omega(X_n)\big]\big|^2 \\
&\leq 8\sum_{\substack{x\cdot\hat{u}=k \\ |x|\leq k}} \mathbb{E}\Bigg[\big|E_x^\omega\big(X_{n-k}\big) - \mathbb{E}\big[E_x^\omega(X_{n-k})\big]\big|^2 \frac{d\mathbb{P}_\infty}{d\mathbb{P}}\bigg|_{\mathfrak{S}_k}\Bigg] + \sum_{\substack{x\cdot\hat{u}=k \\ |x|\leq k}} \mathcal{O}\big(k^2\big),
\end{aligned}$$

where the last line uses the definition of $\mathbb{E}_\infty$, the fact that $E_x^\omega(X_{n-k})$ is measurable with respect to the σ-algebra after time k, and that

$$\big|E_x[X_{n-k}] - E_0[X_n]\big| = \big|x + E_0[X_{n-k} - X_n]\big| \leq |x| + E_0\big|X_{n-k} - X_n\big| \leq 2k.$$

Lastly, we introduce the cone-mixing function by applying Lemma 26. We also use shift-invariance to see that

$$\begin{aligned}
&\mathbb{E}_\infty\big[\big|E_0^\omega(X_n) - \mathbb{E}_\infty\big[E_0^\omega(X_n)\big]\big|^2\big] \\
&\leq 8\sum_{\substack{x\cdot\hat{u}=k \\ |x|\leq k}} \mathbb{E}\big[\big|E_x^\omega\big(X_{n-k}\big) - \mathbb{E}\big[E_x^\omega(X_{n-k})\big]\big|^2\big]\big(1 + \Phi^-(k)\big) + \mathcal{O}\big(k^{d+2}\big)
\end{aligned}$$

$$\leq 8 \sum_{\substack{x\cdot\hat{u}=k \\ |x|\leq k}} \mathbb{E}\big[\big|E_x^\omega\big(X_{n-k}\big) - \mathbb{E}\big[E_x^\omega(X_{n-k})\big]\big|^2\big] + \mathcal{O}\big(k^d n^2 \Phi^-(k)\big) + \mathcal{O}\big(k^{d+2}\big)$$
$$\leq Ck^d \mathbb{E}\big[\big|E_0^\omega\big(X_{n-k}\big) - \mathbb{E}\big[E_0^\omega(X_{n-k})\big]\big|^2\big] + \mathcal{O}\big(k^d n^2 \Phi^-(k)\big) + \mathcal{O}\big(k^{d+2}\big).$$

The second term decays exponentially fast by Assumption 22, so by choosing $0 < \delta < \min\{d^{-1}(1-2\bar{\alpha}), (d+2)^{-1}\}$, the conclusion holds. $\square$

3.3 Step 2: Reduction to Path Intersections

In this section, we reduce our problem of showing a CLT to showing that $E_{0,0}[|X_{[0,n)} \cap \tilde{X}_{[0,n)}|] = \mathcal{O}(n^{2\bar{\alpha}})$ for some $\bar{\alpha} < 1/2$, where $P_{0,0} = \mathbb{E}[P_0^\omega \otimes P_0^\omega]$, and X and $\tilde{X}$ represent independent walkers in the same environment.

We will order sites in C_0^+ as $z_1, z_2, \ldots$ such that for all $i \geq j$, $z_i \cdot \hat{u} \geq z_j \cdot \hat{u}$. As described by Zeitouni in Section 3.1 of [57], consider a weighted coin. We will use this coin to allow the walk to move without using the environment ω for several steps. Let L be a positive integer (the size of the gap). We will flip the coin ξ independently once for every L steps of the walk, resulting in an i.i.d. Bernoulli sequence $\xi_1, \xi_2, \ldots$. This coin will give us the environment for a modified walk, $Y = \{Y_n\}_{n\geq 0}$, as follows: Fix a constant ε satisfying $0 < \varepsilon < \frac{1}{2d}$. With probability $2d\varepsilon$, the coin comes up heads and Y follows a simple symmetric random walk for L steps. Otherwise, the coin comes up tails and Y makes jumps according to the transitions, $\frac{\pi_{xy}(\omega)-\varepsilon}{1-2d\varepsilon}$ for the next L steps. Note that, by choosing $\varepsilon = \frac{\kappa}{2d}$, these probabilities are well defined due to the ellipticity condition (1.5). We will denote the law of the coin ξ by $\mathbf{P}$ and the expectation under $\mathbf{P}$ by $\mathbf{E}$. $P_0^{\omega,\xi}$ is the law of Y_n given ω and ξ.

Lemma 28 *The law of $\{Y_n\}_{n\geq 0}$ under $\int P_0^{\omega,\xi}\,\mathbf{P}(d\xi)$ is the same as the law of $\{X_n\}_{n\geq 0}$ under P_0^ω.*

Proof:

We will show this by direct calculation. It suffices to show that the one-step transitions are the same. Let $\pi_{x,y}^\xi(\omega)$ represent the transition probability from x to y for a fixed coin ξ and environment ω. Then, for $|x-y| = 1$,

$$\int \pi_{x,y}^\xi(\omega)\,\mathbf{P}(d\xi) = 2d\varepsilon \cdot \frac{1}{2d} + (1-2d\varepsilon)\cdot\frac{\pi_{x,y}(\omega)-\varepsilon}{1-2d\varepsilon}$$
$$= \pi_{x,y}(\omega),$$

so we are done. $\square$

Fix positive integers n and L. Define a stopping time, τ as follows:

$$\tau = \inf\{i > 0 : \xi_i = \text{heads}\}, \tag{3.5}$$

the first time the coin comes up heads Note that τ is a geometric random variable with probability of success κ, so $\mathbf{E}[\tau] = \kappa^{-1}$.

Proposition 29 *Assume that there exists an $\bar{\alpha} < 1/2$ such that $E_{0,0}\big[\big|X_{[0,n)} \cap \tilde{X}_{[0,n)}\big|\big] = \mathcal{O}(n^{2\bar{\alpha}})$, where X and $\tilde{X}$ are independent walks in the same environment. Then there exists an $\alpha < 1/2$ such that*

$$\mathbb{E}\Big[\big|E_0^\omega(X_n) - E_0[X_n]\big|^2\Big] = \mathcal{O}\big(n^{2\alpha}\big). \tag{3.6}$$

Proof:

Define the sets $A_j = \{z_i : i \le j\}$ and $\mathcal{H}_j = \{z_i : i > j\} = C_0^+ \backslash A_j$. Let $\mathbb{P}^{\omega_B}$ represent the regular conditional probability $\mathbb{P}$ given a fixed ω_B.

For n fixed, $\mathbb{E}[E_0^\omega(X_n) \,|\, \mathfrak{S}_{A_j}]$ is a martingale, so we can calculate the expression on line (3.6) by

$$\begin{aligned}
\mathbb{E}\Big[\big|E_0^\omega(X_n) - E_0[X_n]\big|^2\Big] &= \sum_j \mathbb{E}\Big[\big|\mathbb{E}\big[E_0^\omega(X_n) \,\big|\, \mathfrak{S}_{A_j}\big] - \mathbb{E}\big[E_0^\omega(X_n) \,\big|\, \mathfrak{S}_{A_{j-1}}\big]\big|^2\Big] \\
= \sum_j \int \bigg| \int E_0^\omega(X_n)\, \mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j}) &- \int E_0^{\tilde{\omega}}(X_n)\, \mathbb{P}^{\omega_{A_{j-1}}}(d\omega_{\mathcal{H}_j} d\tilde{\omega}_{z_j}) \bigg|^2 \mathbb{P}(d\omega_{A_j}) \\
= \sum_j \int \bigg| \iint E_0^\omega(X_n)\, \mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j})\, \mathbb{P}^{\omega_{A_{j-1}}}(d\tilde{\omega}_{z_j}) & \\
- \iint E_0^{\tilde{\omega}}(X_n)\, \mathbb{P}^{\omega_{A_{j-1}}\tilde{\omega}_{z_j}}(d\omega_{\mathcal{H}_j})\, \mathbb{P}^{\omega_{A_{j-1}}}(d\tilde{\omega}_{z_j}) \bigg|^2 \mathbb{P}(d\omega_{A_j}). &
\end{aligned}$$

We can then apply Jensen's inequality to bound line (3.6) by

$$\begin{aligned}
&\mathbb{E}\Big[\big|E_0^\omega(X_n) - E_0[X_n]\big|^2\Big] \\
&\le \sum_j \iiint \bigg| \int E_0^\omega(X_n)\, \mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j}) \\
&\quad - \int E_0^{\tilde{\omega}}(X_n)\, \mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j}) \bigg|^2 \mathbb{P}^{\omega_{A_{j-1}}}(d\tilde{\omega}_{z_j})\, \mathbb{P}^{\omega_{A_{j-1}}}(d\omega_{z_j})\, \mathbb{P}(d\omega_{A_{j-1}}) \\
&= \sum_j \iiint \bigg| \iint E_0^{\omega,\xi}(Y_n)\, \mathbf{P}(d\xi)\, \mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j}) \\
&\quad - \iint E_0^{\tilde{\omega},\xi}(Y_n)\, \mathbf{P}(d\xi)\, \mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j}) \bigg|^2 \mathbb{P}^{\omega_{A_{j-1}}}(d\tilde{\omega}_{z_j})\, \mathbb{P}^{\omega_{A_{j-1}}}(d\omega_{z_j})\, \mathbb{P}(d\omega_{A_{j-1}}).
\end{aligned}$$

Here, $\tilde{\omega}_x = \omega_x$ for $x \neq z_j$.

Fix $z_j \in C_0^+$, and let $k = z_j \cdot \hat{u}$. Let I represent the above integrand. Fix $\delta > 0$, and let $0 < \ell = n^\delta < n$. In order to bound I, we will consider the sites near z_j and far from z_j separately by computing

$$\begin{aligned}
I &= \left| \iint E_0^{\omega,\xi}(Y_n)\,\mathbf{P}(d\xi)\,\mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j}) - \iint E_0^{\tilde{\omega},\xi}(Y_n)\,\mathbf{P}(d\xi)\,\mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j}) \right| \\
&= \Bigg| \iint \sum_{\substack{x\cdot\hat{u}=k \\ |x|\le k}} P_0^{\omega,\xi}\{Y_k = x\} E_x^{\omega,\xi}(Y_{n-k})\,\mathbf{P}(d\xi)\,\mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j}) \\
&\quad - \iint \sum_{\substack{x\cdot\hat{u}=k \\ |x|\le k}} P_0^{\omega,\xi}\{Y_k = x\} E_x^{\tilde{\omega},\xi}(Y_{n-k})\,\mathbf{P}(d\xi)\,\mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j}) \Bigg| \\
&= \Bigg| \int \sum_{\substack{x\cdot\hat{u}=k \\ |x|\le k}} P_0^{\omega,\xi}\{Y_k = x\}\,\mathbf{P}(d\xi) \cdot \iint E_x^{\omega,\xi}(Y_{n-k})\,\mathbf{P}(d\xi)\,\mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j}) \\
&\quad - \int \sum_{\substack{x\cdot\hat{u}=k \\ |x|\le k}} P_0^{\omega,\xi}\{Y_k = x\}\,\mathbf{P}(d\xi) \cdot \iint E_x^{\tilde{\omega},\xi}(Y_{n-k})\,\mathbf{P}(d\xi)\,\mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j}) \Bigg| \\
&\le \int \sum_{\substack{x\cdot\hat{u}=k \\ |x|\le k}} P_0^{\omega,\xi}\{Y_k = x\}\mathbf{P}(d\xi) \Bigg| \iint E_x^{\omega,\xi}(Y_{n-k})\,\mathbf{P}(d\xi)\,\mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j}) \\
&\quad - \iint E_x^{\tilde{\omega},\xi}(Y_{n-k})\,\mathbf{P}(d\xi)\,\mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j}) \Bigg| \\
&= \sum_{\substack{x\cdot\hat{u}=k \\ |x-z_j|\le \ell}} P_0^{\omega}\{X_k = x\} \Bigg| \iint E_x^{\omega,\xi}(Y_{n-k})\,\mathbf{P}(d\xi)\,\mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j}) \\
&\quad - \iint E_x^{\tilde{\omega},\xi}(Y_{n-k})\,\mathbf{P}(d\xi)\,\mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j}) \Bigg| \qquad (3.7) \\
&\quad + \sum_{\substack{x\cdot\hat{u}=k \\ |x|\le k \\ \ell<|x-z_j|}} P_0^{\omega}\{X_k = x\} \left| \int E_x^{\omega}(X_{n-k})\,\mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j}) - \int E_x^{\omega}(X_{n-k})\,\mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j}) \right|. \qquad (3.8)
\end{aligned}$$

We will bound each of lines (3.7) and (3.8) separately. We will first consider line (3.7), where the walkers go through a point on level k close to z_j. We proceed by restarting the walks, Y in environment ω, ξ, and Y in environment $\tilde{\omega}, \xi$, at time $L\tau$ (which only depends on ξ), taking into special consideration those values of τ which are larger than $\frac{n-k}{L}$.

$$\begin{aligned}
&\sum_{\substack{x\cdot\hat{u}=k \\ |x-z_j|\le \ell}} P_0^{\omega}\{X_k = x\} \Bigg| \iint E_x^{\omega,\xi}(Y_{n-k})\,\mathbf{P}(d\xi)\,\mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j}) \\
&\quad - \iint E_x^{\tilde{\omega},\xi}(Y_{n-k})\,\mathbf{P}(d\xi)\,\mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j}) \Bigg|
\end{aligned}$$

$$\le \sum_{\substack{x\cdot\hat{u}=k\\|x-z_j|\le\ell}} P_0^\omega\{X_k=x\}\cdot$$

$$\Bigg|\iint \mathbb{1}\Big\{\tau\le\frac{n-k}{L}\Big\}\sum_{\substack{y\cdot\hat{u}=k+\tau L\\|y-x|\le\tau L}} P_x^{\omega,\xi}\{Y_{\tau L}=y\}E_y^{\omega,\xi}(Y_{n-k-\tau L})\,\mathbf{P}(d\xi)\,\mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j})$$

$$-\iint \mathbb{1}\Big\{\tau\le\frac{n-k}{L}\Big\}\sum_{\substack{y\cdot\hat{u}=k+\tau L\\|y-x|\le\tau L}} P_x^{\tilde{\omega},\xi}\{Y_{\tau L}=y\}E_y^{\omega,\xi}(Y_{n-k-\tau L})\,\mathbf{P}(d\xi)\,\mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j})\Bigg|$$

$$+\sum_{\substack{x\cdot\hat{u}=k\\|x-z_j|\le\ell}} P_0^\omega\{X_k=x\}\Bigg|\iint E_x^{\omega,\xi}(Y_{n-k})\mathbb{1}\{\tau L>n-k\}\,\mathbf{P}(d\xi)\,\mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j})$$

$$-\iint E_x^{\tilde{\omega},\xi}(Y_{n-k})\mathbb{1}\{\tau L>n-k\}\,\mathbf{P}(d\xi)\,\mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j})\Bigg|.$$

To bound the second sum in the above equation, note that

$$\begin{aligned}
&\Bigg|\iint E_x^{\omega,\xi}(Y_{n-k})\mathbb{1}\{\tau L>n-k\}\,\mathbf{P}(d\xi)\,\mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j})\\
&\quad-\iint E_x^{\tilde{\omega},\xi}(Y_{n-k})\mathbb{1}\{\tau L>n-k\}\,\mathbf{P}(d\xi)\,\mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j})\Bigg|\\
&=\Bigg|\iint E_{x,x}^{\bar{\omega},\hat{\omega},\xi}\big(Y_{n-k}-\bar{Y}_{n-k}\big)\mathbb{1}\{\tau L>n-k\}\,\mathbf{P}(d\xi)\,\mathbb{P}^{\omega_{A_j}}(d\bar{\omega}_{\mathcal{H}_j})\,\mathbb{P}^{\tilde{\omega}_{A_j}}(d\hat{\omega}_{\mathcal{H}_j})\Bigg|\\
&\le 2(n-k)\mathbf{P}\{\tau L>n-k\},
\end{aligned}$$

where Y and $\bar{Y}$ represent walks in environments $\hat{\omega},\xi$ and $\bar{\omega},\xi$, respectively.

Next, we will apply Fubini's theorem to see that line (3.7) is bounded above by

$$\sum_{\substack{x\cdot\hat{u}=k\\|x-z_j|\le\ell}} P_0^\omega\{X_k=x\}\Bigg|\iint E_x^{\omega,\xi}(Y_{n-k})\,\mathbf{P}(d\xi)\,\mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j})$$

$$-\iint E_x^{\tilde{\omega},\xi}(Y_{n-k})\,\mathbf{P}(d\xi)\,\mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j})\Bigg|$$

$$\le \sum_{\substack{x\cdot\hat{u}=k\\|x-z_j|\le\ell}} P_0^\omega\{X_k=x\}\cdot$$

$$\Bigg(\Bigg|\int\sum_{1\le m\le\frac{n-k}{L}}\int \mathbb{1}\{\tau=m\}\sum_{\substack{y\cdot\hat{u}=k+mL\\|y-x|\le mL}} P_x^{\omega,\xi}\{Y_{mL}=y\}E_y^{\omega,\xi}(Y_{n-k-mL})\,\mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j})\,\mathbf{P}(d\xi)$$

$$-\int\sum_{1\le m\le\frac{n-k}{L}}\int \mathbb{1}\{\tau=m\}\sum_{\substack{y\cdot\hat{u}=k+mL\\|y-x|\le mL}} P_x^{\tilde{\omega},\xi}\{Y_{mL}=y\}E_y^{\omega,\xi}(Y_{n-k-mL})\,\mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j})\,\mathbf{P}(d\xi)\Bigg|$$

$$+\,2(n-k)\mathbf{P}\{\tau L>n-k\}\Bigg)$$

$$
\begin{aligned}
&\leq \sum_{\substack{x\cdot\hat{u}=k\\|x-z_j|\leq\ell}} P_0^\omega\{X_k=x\}\cdot \\
&\Bigg(\Bigg|\int \sum_{1\leq m\leq\frac{n-k}{L}} \mathbb{1}\{\tau=m\}\Bigg(\int \sum_{\substack{y\cdot\hat{u}=k+mL\\|y-x|\leq mL}} P_x^{\omega,\xi}\{Y_{mL}=y\}E_y^{\omega,\xi}(Y_{n-k-mL})\,\mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j}) \qquad (3.9)\\
&\quad-\int \sum_{\substack{y\cdot\hat{u}=k+mL\\|y-x|\leq mL}} P_x^{\tilde{\omega},\xi}\{Y_{mL}=y\}E_y^{\omega,\xi}(Y_{n-k-mL})\,\mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j})\Bigg)\,\mathbf{P}(d\xi)\Bigg| \\
&\quad+2L\mathbf{E}\big[\tau\mathbb{1}\{\tau L>n-k\}\big]\Bigg). \qquad (3.10)
\end{aligned}
$$

Define $B=B_m=\{w\in C_x^+\backslash A_j : k\leq w\cdot\hat{u}\leq k+(m-1)L\}$, the sites in C_x^+ before the L levels where the coin was heads (i.e. the sites where the walkers used the environment). We will now aim to bound the integrand of the outer integral in line (3.9) by conditioning on ω_B. Note that x, $\tau=m$ and ξ will be fixed in the following calculation.

$$
\begin{aligned}
&\Bigg|\int \sum_{\substack{y\cdot\hat{u}=k+mL\\|y-x|\leq mL}} P_x^{\omega,\xi}\{Y_{mL}=y\}E_y^{\omega,\xi}(Y_{n-k-mL})\,\mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j}) \\
&\quad-\int \sum_{\substack{y\cdot\hat{u}=k+mL\\|y-x|\leq mL}} P_x^{\tilde{\omega},\xi}\{Y_{mL}=y\}E_y^{\omega,\xi}(Y_{n-k-mL})\,\mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j})\Bigg| \\
&=\Bigg|\sum_{\substack{y\cdot\hat{u}=k+mL\\|y-x|\leq mL}} \iint P_x^{\omega,\xi}\{Y_{mL}=y\}E_y^{\omega,\xi}(Y_{n-k-mL})\,\mathbb{P}^{\omega_B,\omega_{A_j}}(d\omega_{C_x^+})\,\mathbb{P}^{\omega_{A_j}}(d\omega_B) \\
&\quad-\sum_{\substack{y\cdot\hat{u}=k+mL\\|y-x|\leq mL}} \iint P_x^{\tilde{\omega},\xi}\{Y_{mL}=y\}E_y^{\omega,\xi}(Y_{n-k-mL})\,\mathbb{P}^{\omega_B,\tilde{\omega}_{A_j}}(d\omega_{C_x^+})\,\mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_B)\Bigg| \\
&=\Bigg|\sum_{\substack{y\cdot\hat{u}=k+mL\\|y-x|\leq mL}} \int P_x^{\omega,\xi}\{Y_{mL}=y\}\bigg(\int E_y^{\omega,\xi}(Y_{n-k-mL})\,\mathbb{P}^{\omega_B,\omega_{A_j}}(d\omega_{C_y^+})\bigg)\,\mathbb{P}^{\omega_{A_j}}(d\omega_B) \\
&\quad-\sum_{\substack{y\cdot\hat{u}=k+mL\\|y-x|\leq mL}} \int P_x^{\tilde{\omega},\xi}\{Y_{mL}=y\}\bigg(\int E_y^{\omega,\xi}(Y_{n-k-mL})\,\mathbb{P}^{\omega_B,\tilde{\omega}_{A_j}}(d\omega_{C_y^+})\bigg)\,\mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_B)\Bigg|
\end{aligned}
$$

where the last line uses the fact that $P_x^{\omega,\xi}\{Y_{mL}=y\}$ and $P_x^{\tilde{\omega},\xi}\{Y_{mL}=y\}$ are measurable with respect to $\sigma(\omega_B,\omega_{A_j})$ and $\sigma(\omega_B,\tilde{\omega}_{A_j})$, respectively.

Since the walkers have not used the environment ω for at least L steps, the cone-mixing function Φ^+ defined in (3.1) can now be applied to bound the integrand on line (3.9) by

$$\begin{aligned}
&\Big| \int \sum_{\substack{y\cdot\hat{u}=k+mL\\|y-x|\le mL}} P_x^{\omega,\xi}\{Y_{mL}=y\} E_y^{\omega,\xi}(Y_{n-k-mL})\,\mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j}) \\
&\quad - \int \sum_{\substack{y\cdot\hat{u}=k+mL\\|y-x|\le mL}} P_x^{\tilde\omega,\xi}\{Y_{mL}=y\} E_y^{\omega,\xi}(Y_{n-k-mL})\,\mathbb{P}^{\tilde\omega_{A_j}}(d\omega_{\mathcal{H}_j}) \Big| \\
&\le 2n\Phi^+(L) + \Big| \sum_{\substack{y\cdot\hat{u}=k+mL\\|y-x|\le mL}} \int P_x^{\omega,\xi}\{Y_{mL}=y\} \Big(\int E_y^{\omega,\xi}(Y_{n-k-mL})\,\mathbb{P}(d\omega_{C_y^+}) \Big)\, \mathbb{P}^{\omega_{A_j}}(d\omega_B) \\
&\quad - \sum_{\substack{y\cdot\hat{u}=k+mL\\|y-x|\le mL}} \int P_x^{\tilde\omega,\xi}\{Y_{mL}=y\} \Big(\int E_y^{\omega,\xi}(Y_{n-k-mL})\,\mathbb{P}(d\omega_{C_y^+}) \Big)\, \mathbb{P}^{\tilde\omega_{A_j}}(d\omega_B) \Big| \\
&= 2n\Phi^+(L) + \Big| \sum_{\substack{y\cdot\hat{u}=k+mL\\|y-x|\le mL}} E_y^{\xi}(Y_{n-k-mL}) \int P_x^{\omega,\xi}\{Y_{mL}=y\}\,\mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j}) \\
&\quad - \sum_{\substack{y\cdot\hat{u}=k+mL\\|y-x|\le mL}} E_y^{\xi}(Y_{n-k-mL}) \int P_x^{\tilde\omega,\xi}\{Y_{mL}=y\}\,\mathbb{P}^{\tilde\omega_{A_j}}(d\omega_{\mathcal{H}_j}) \Big| \\
&\le 2n\Phi^+(L) + \max_{\substack{y\cdot\hat{u}=y'\cdot\hat{u}=k+mL\\|y-x|\le mL\\|y'-x|\le mL}} \big| E_y^{\xi}(Y_{n-k-mL}) - E_{y'}^{\xi}(Y_{n-k-mL}) \big| \\
&= 2n\Phi^+(L) + \max_{\substack{y\cdot\hat{u}=y'\cdot\hat{u}=k+mL\\|y-x|\le mL\\|y'-x|\le mL}} \big| y-y' \big| \le 2n\Phi^+(L) + 2mL.
\end{aligned}$$

Integrating out the coin ξ, we then get that line (3.9) is bounded above by

$$2n\Phi^+(L) + 2L\mathbf{E}\big[\tau\mathbb{1}\{\tau L \le n-k\}\big]. \tag{3.11}$$

Combining lines (3.10) and (3.11), we conclude that line (3.7) is bounded above by

$$(2n\Phi^+(L) + 2L\mathbf{E}[\tau]) \sum_{\substack{x\cdot\hat{u}=k\\|x-z_j|\le\ell}} P_0^{\omega}\{X_k=x\}. \tag{3.12}$$

Let us now return to bounding line (3.8), where the walkers pass at least ℓ units away from the altered site, z_j. Since the walks do not pass near the altered site, we can keep them coupled. However, the walks are averaged against different measures. Fix x on level k such that $|x-z_j|>\ell$ and $|x|\le k$. Noting that $E_x^{\omega}(X_{n-k})$ is $\mathfrak{S}_{C_x^+}$-measurable and $C_x^+ \subset \mathcal{H}_j$, we see that

$$\Big| \int E_x^{\omega}(X_{n-k})\,\mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j}) - \int E_x^{\omega}(X_{n-k})\,\mathbb{P}^{\tilde\omega_{A_j}}(d\omega_{\mathcal{H}_j}) \Big|$$

$$= \left| \int E_x^\omega(X_{n-k})\, \mathbb{P}^{\omega_{A_{j-1}}\omega_{z_j}}(d\omega_{C_x^+}) - \int E_x^\omega(X_{n-k})\, \mathbb{P}^{\omega_{A_{j-1}}\tilde{\omega}_{z_j}}(d\omega_{C_x^+}) \right|$$
$$\le n\Psi(\ell)$$

by using the spatial mixing function defined in (3.2) in the last line.

Then, going back to the original calculation, and combining the bounds for lines (3.7) and (3.8), we observe that

$$\begin{aligned}
&\mathbb{E}\Big[\big|E_0^\omega(X_n) - E_0[X_n]\big|^2\Big] \\
&\le \sum_j \iiint \Bigg[\sum_{\substack{x\cdot\hat{u}=k \\ |x-z_j|\le \ell}} P_0^\omega\{X_k = x\} \Bigg| \iint E_x^{\omega,\xi}(Y_{n-k})\, \mathbf{P}(d\xi)\, \mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j}) \\
&\quad - \iint E_x^{\tilde{\omega},\xi}(Y_{n-k})\, \mathbf{P}(d\xi)\, \mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j}) \Bigg| \\
&\quad + \sum_{\substack{x\cdot\hat{u}=k \\ |x|\le k \\ \ell<|x-z_j|}} P_0^\omega\{X_k = x\} \Bigg| \iint E_x^\omega(X_{n-k})\, \mathbb{P}^{\omega_{A_j}}(d\omega_{\mathcal{H}_j}) \\
&\quad - \iint E_x^\omega(X_{n-k})\, \mathbb{P}^{\tilde{\omega}_{A_j}}(d\omega_{\mathcal{H}_j}) \Bigg| \Bigg]^2 \mathbb{P}^{\omega_{A_{j-1}}}(d\tilde{\omega}_{z_j})\, \mathbb{P}^{\omega_{A_{j-1}}}(d\omega_{z_j})\, \mathbb{P}(d\omega_{A_{j-1}}) \\
&\le \sum_j \iiint \Bigg[\big(2n\Phi^+(L) + 2L\mathbf{E}[\tau]\big) \sum_{\substack{x\cdot\hat{u}=k \\ |x-z_j|\le \ell}} P_0^\omega\{X_k = x\} \\
&\quad + n\Psi(\ell) \sum_{\substack{x\cdot\hat{u}=k \\ |x|\le k \\ \ell<|x-z_j|}} P_0^\omega\{X_k = x\} \Bigg]^2 \mathbb{P}^{\omega_{A_{j-1}}}(d\tilde{\omega}_{z_j})\, \mathbb{P}^{\omega_{A_{j-1}}}(d\omega_{z_j})\, \mathbb{P}(d\omega_{A_{j-1}}).
\end{aligned}$$

Using Jensen's inequality, we get that

$$\begin{aligned}
&\mathbb{E}\Big[\big|E_0^\omega(X_n) - E_0[X_n]\big|^2\Big] \\
&\le C \sum_j \iiint \Bigg[\big(2n\Phi^+(L) + 2L\mathbf{E}[\tau]\big)^2 \ell^d \sum_{\substack{x\cdot\hat{u}=k \\ |x-z_j|\le \ell}} P_0^\omega\{X_k = x\}^2 \\
&\quad + n^2\Psi(\ell)^2 n^d \sum_{\substack{x\cdot\hat{u}=k \\ |x|\le k \\ \ell<|x-z_j|}} P_0^\omega\{X_k = x\}^2 \Bigg] \mathbb{P}^{\omega_{A_{j-1}}}(d\tilde{\omega}_{z_j})\, \mathbb{P}^{\omega_{A_{j-1}}}(d\omega_{z_j})\, \mathbb{P}(d\omega_{A_{j-1}}) \\
&\le C\big(\ell^{2d} n^2 \Phi^+(L)^2 + \ell^{2d} L^2 \mathbf{E}[\tau]^2\big) \sum_j \mathbb{E}\big[P_0^\omega\{X_k = z_j\}^2\big] + \mathcal{O}\big(n^{2d+3}\Psi(\ell)^2\big) \\
&= C\big(\ell^{2d} n^2 \Phi^+(L)^2 + \ell^{2d} L^2 \kappa^{-2}\big) E_{0,0}\big[\big|X_{[0,n)} \cap \tilde{X}_{[0,n)}\big|\big] + \mathcal{O}\big(n^{2d+3}\Psi(\ell)^2\big),
\end{aligned}$$

where X and $\tilde{X}$ are independent walks in the same environment. The error term is exponentially decreasing, so by choosing $\ell = n^\delta$ with $0 < \delta < (1-2\bar{\alpha})/2d$, and $L = \beta \log n$ with $\beta > (2d\delta + 2\bar{\alpha} + 1)/2\lambda$, we are done. □

3.4 Step 3: From Two RWREs to One Markov Chain

In this section we will start bounding $E_{0,0}[|X_{[0,n)} \cap \tilde{X}_{[0,n)}|]$, the expected number of intersections of two independent walks in a common environment. We first show that the difference between the two walks is "almost" a Markov chain. Throughout this section, we will let $B_\rho = [-\rho, \rho]^d$, the ball of radius ρ.

Recall the definition of the coin ξ from Section 3.3. From now on, we will abuse notation and refer to X for the walk in both the "regular" and "coin" environments, depending on context. Fix a positive integer L and let X and $\tilde{X}$ be independent walks in the same environment using the same coin. Then, we redefine $Y = \{Y_i\}_{i\geq 0}$ by $Y_i = X_{\tau_i L} - \tilde{X}_{\tau_i L}$, the distance between the two paths after not using the environment for L steps. Note that Y depends on the choice of L. Let τ_i for $i \geq 0$ be defined as follows: $\tau_0 = 0$ and $\tau_i = \inf\{n > \tau_{i-1} : \xi_n = \text{heads}\}$, the i^{th} time that the coin has come up heads.

Lemma 30 *For $L = \beta \log n$ and $r = n^\varepsilon$ with $\varepsilon, \beta > 0$, we have that $E_{0,0}[|X_{[0,n)} \cap \tilde{X}_{[0,n)}|] \leq CLrE_{0,0}\left[\sum_{k=0}^{n-1} \mathbb{1}\{Y_k \in B_{Lr}\}\right]$ for sufficiently large n.*

Proof:
We will bound $E_{0,0}[|X_{[0,n)} \cap \tilde{X}_{[0,n)}|]$ above by considering how long it takes for each τ_i to occur. Using that $\tau_{n-1} \geq n-1$,

$$\begin{aligned}
E_{0,0}\left[|X_{[0,n)} \cap \tilde{X}_{[0,n)}|\right] &= E_{0,0}\left[\sum_{i=0}^{n-1} \mathbb{1}\{X_i = \tilde{X}_i\}\right] \\
&\leq \sum_{i=0}^{n-1} E_{0,0}\left[\sum_{k=\tau_i}^{\tau_{i+1}-1} \mathbb{1}\{X_i = \tilde{X}_i\}\right] \\
&\leq \sum_{i=0}^{n-1} E_{0,0}\left[\mathbb{1}\left\{\tau_{i+1} - \tau_i \leq \frac{r}{2}\right\}\mathbb{1}\{|Y_i| \leq Lr\} \cdot \frac{Lr}{2}\right. \\
&\quad \left. + \mathbb{1}\left\{\tau_{i+1} - \tau_i > \frac{r}{2}\right\} \cdot L(\tau_{i+1} - \tau_i)\right].
\end{aligned}$$

In other words, if the renewal happens quickly (within time $r/2$), not many intersections happen even if $Y_i \in B_{Lr}$. Also, note that if the renewal happens within time $r/2$ and

$Y_i \notin B_{Lr}$, no intersections will happen before time $L\tau_{i+1}$. However, when the renewal takes longer than time $r/2$, even though many intersections can occur, this happens with small probability. Using that the $\{\tau_{i+1} - \tau_i\}$'s are i.i.d., we continue bounding the above by

$$E_{0,0}\big[|X_{[0,n)} \cap \tilde{X}_{[0,n)}|\big] \le LrE_{0,0}\Bigg[\sum_{k=0}^{n-1} \mathbb{1}\{Y_k \in B_{Lr}\}\Bigg] + Ln\mathbf{E}\Bigg[\tau_1 \mathbb{1}\Big\{\tau_1 > \frac{r}{2}\Big\}\Bigg].$$

Now we will consider the error term for when τ_1 is large. Since geometric random variables are memoryless,

$$\begin{aligned} Ln\mathbf{E}\Bigg[\tau_1 \mathbb{1}\Big\{\tau_1 > \frac{r}{2}\Big\}\Bigg] &= Ln\bigg(\mathbf{E}[\tau_1] + \frac{r}{2}\bigg)\mathbf{P}\Big\{\tau_1 > \frac{r}{2}\Big\} \\ &= Ln\bigg(\kappa^{-1} + \frac{r}{2}\bigg)(1-\kappa)^{r/2} \\ &\le Ln\bigg(\kappa^{-1} + \frac{r}{2}\bigg)e^{-r\kappa/2} \\ &= \beta n \log n\bigg[\kappa^{-1} + \frac{n^{\varepsilon}}{2}\bigg]e^{-\kappa n^{\varepsilon}/2}. \end{aligned}$$

Since $\beta n \log n[\kappa^{-1} + \frac{n^\varepsilon}{2}]$ is polynomially increasing, and $e^{-\kappa n^\varepsilon/2}$ is exponentially decreasing, this quantity is small for sufficiently large n. Therefore, we are done. $\square$

Let $\bar{Y}_k$ represent the Markov chain starting at y with the transition probabilities $P_y\{\bar{Y}_{k+1} = z \,|\, \bar{Y}_k = x\} = E_{0,x}[\mathbb{1}\{Y_1 = z\}]$.

Proposition 31 *Let $y_1, \dots, y_n \in \mathbb{Z}^{d+1}$. Then,*

$$\begin{aligned} (1-\Phi^+(L))^n P_y\big\{\bar{Y}_k = y_k \textit{ for } k = 1,\dots,n\big\} &\le P_y\big\{Y_k = y_k \textit{ for } k = 1,\dots,n\big\} \\ &\le (1+\Phi^+(L))^n P_y\big\{\bar{Y}_k = y_k \textit{ for } k = 1,\dots,n\big\}. \end{aligned}$$

Proof:

We will calculate this directly. For $x, y \in \mathbb{Z}^{d+1}$, define $C^+_{x,y} = C^+_x \cup C^+_y$ and $C^-_{x,y} = C^-_x \cup C^-_y$.

$$\begin{aligned} &P_{0,y}\big\{Y_k = y_k \text{ for } k = 1,\dots,n\big\} \\ &= \sum_{x_{1,n},\tilde{x}_{1,n}} \iiint P^{\omega,\xi}_{0,y}\big\{X_{\tau_1 L} = x_1, \tilde{X}_{\tau_1 L} = \tilde{x}_1\big\} \times \\ &\quad \prod_{i=2}^{n} P^{\omega,\xi}_{x_{i-1},\tilde{x}_{i-1}}\big\{X_{\tau_i L} = x_i, \tilde{X}_{\tau_i L} = \tilde{x}_i\big\}\, \mathbf{P}(d\xi)\, \mathbb{P}^{\omega_{C^+_{x_2,\tilde{x}_2}}}(d\omega_{C^-_{x_1,\tilde{x}_1}})\, \mathbb{P}(d\omega_{C^+_{x_2,\tilde{x}_2}}) \\ &= (1-\Phi^+(L)) \sum_{x_{1,n},\tilde{x}_{1,n}} P_{0,y}\big\{X_{\tau_1 L} = x_1, \tilde{X}_{\tau_1 L} = \tilde{x}_1\big\} \times \\ &\quad P_{x_1,\tilde{x}_1}\big\{X_{\tau_1 L} = x_2, \tilde{X}_{\tau_1 L} = \tilde{x}_2, \dots, X_{\tau_{n-1} L} = x_n, \tilde{X}_{\tau_{n-1} L} = \tilde{x}_n\big\} \end{aligned}$$

$$= (1 - \Phi^+(L))P_y\big\{\bar{Y}_1 = y_1\big\}P_{y_1}\big\{Y_1 = y_2, \dots, Y_{n-1} = y_n\big\}$$
$$\geq \cdots \geq (1 - \Phi^+(L))^n P_y\big\{\bar{Y}_k = y_k \text{ for } k = 1, \dots, n\big\}.$$

The sum in the above computation was over paths $x_{1,n}, \tilde{x}_{1,n}$ in $\mathbb{Z}^{d+1}$, such that $x_0 = 0$, $\tilde{x}_{1,n} - x_{1,n} = y_{1,n}$, and $x_i \cdot \hat{u} = \tilde{x}_i \cdot \hat{u} = y_i \cdot \hat{u}$. Similarly, the upper bound also holds. □

Proposition 32 *The averaged expected number of times Y and $\bar{Y}$ are near zero, $E_{0,0}\big[\sum_{k=0}^{n-1} \mathbb{1}\{Y_k \in B_{Lr}\}\big]$ and $E_0\big[\sum_{k=0}^{n-1} \mathbb{1}\{\bar{Y}_k \in B_{Lr}\}\big]$, are of the same order of magnitude if $L \geq \lambda^{-1} \log n$, where λ is from the cone-mixing condition in Assumption 22.*

Proof:

By Proposition 31,

$$E_{0,0}\left[\sum_{k=0}^{n-1} \mathbb{1}\{Y_k \in B_{Lr}\}\right] = \sum_{y=(0,y_1,\dots,y_{n-1})\in\mathbb{Z}^d} |y \cap B_{Lr}|\, P_{0,0}\{Y = y\}$$
$$\leq (1 + \Phi^+(L))^n \sum_y |y \cap B_{Lr}|\, P_0\{\bar{Y} = y\}$$
$$= (1 + \Phi^+(L))^n E_0\left[\sum_{k=0}^{n-1} \mathbb{1}\{\bar{Y}_k \in B_{Lr}\}\right].$$

Similarly,

$$E_{0,0}\left[\sum_{k=0}^{n-1} \mathbb{1}\{Y_k \in B_{Lr}\}\right] \geq (1 - \Phi^+(L))^n E_0\left[\sum_{k=0}^{n-1} \mathbb{1}\{\bar{Y}_k \in B_{Lr}\}\right].$$

Next, by taking $L \geq \lambda^{-1} \log n$, we get that

$$c^{-1} E_0\left[\sum_{k=0}^{n-1} \mathbb{1}\{\bar{Y}_k \in B_{Lr}\}\right] \leq E_{0,0}\left[\sum_{k=0}^{n-1} \mathbb{1}\{Y_k \in B_{Lr}\}\right] \leq c E_0\left[\sum_{k=0}^{n-1} \mathbb{1}\{\bar{Y}_k \in B_{Lr}\}\right],$$

where c is a constant. Then, we see that the result holds. □

As a result of Proposition 32 and Lemma 30, it will suffice to show that there exist $\varepsilon > 0$, $\beta > \lambda^{-1}$ and $\bar{\alpha} < 1/2$ such that $E_0\big[\sum_{k=0}^{n-1} \mathbb{1}\{\bar{Y}_k \in B_{Lr}\}\big] = \mathcal{O}(n^{2\bar{\alpha}})$ for $L = \beta \log n$ and $r = n^{\varepsilon}$ in order to prove a CLT.

3.5 Step 4: From Markov Chain to Random Walk and Back

Now we will work out a coupling between $\bar{Y}$ and $\bar{\bar{Y}}$, where $\bar{\bar{Y}} = (\bar{\bar{Y}}_k)_{k\geq 0}$ is the symmetric random walk with transitions $p(x, y) = \mathbf{E}\big[(E_0^{\xi} \otimes E_x^{\xi})[\mathbb{1}\{Y_1 = y\}]\big]$.

Lemma 33 $\bar{\bar{Y}}$ *with transitions* $p(x,y) = \mathbf{E}\big[(E_0^\xi \otimes E_x^\xi)[\mathbb{1}\{Y_1 = y\}]\big]$ *is a symmetric random walk.*

Proof:

From the definition of Y, $Y_i = X_{\tau_i L} - \tilde{X}_{\tau_i L}$, so

$$\begin{aligned}
p(x,y) &= \mathbf{E}\big[(E_0^\xi \otimes E_x^\xi)[\mathbb{1}\{Y_1 = y\}]\big] \\
&= \mathbf{E}\big[(E_0^\xi \otimes E_x^\xi)[\mathbb{1}\{X_{\tau_i L} - \tilde{X}_{\tau_i L} = y\}]\big] \\
&= \mathbf{E}\big[(E_0^\xi \otimes E_x^\xi)[\mathbb{1}\{X_{\tau_i L} - (\tilde{X}_{\tau_i L} + x) = y - x\}]\big] \\
&= \mathbf{E}\big[(E_0^\xi \otimes E_0^\xi)[\mathbb{1}\{X_{\tau_i L} - \tilde{X}_{\tau_i L} = y - x\}]\big] \\
&= p(0, y - x).
\end{aligned}$$

As a result, the transition probabilities meet the requirement for those of a random walk.

Since the naming of X and $\tilde{X}$ was arbitrary, $\bar{\bar{Y}}$ is also symmetric. □

Lemma 34 *If L is large, $\bar{Y}$ and $\bar{\bar{Y}}$ can be coupled such that $P_{x,x}\{\bar{Y}_1 \neq \bar{\bar{Y}}_1\} \leq Cb^{-|x|}$ where $b = b(L) = (1-\kappa)^{-1/(4L)} > 1$. C is a constant independent of L.*

Proof:

We will use a technique from [3] to couple $\bar{Y}$ and $\bar{\bar{Y}}$. Let X and $\tilde{X}$ represent independent walks in the same environment. Fix the coin environment ξ. Since τ and $\tilde{\tau}$ only depend on ξ, $\tau_1 = \tilde{\tau}_1$. Let $\tau_1 L = \tilde{\tau}_1 L = n$ and let $x_{0,n}$ and $\tilde{x}_{0,n}$ be two nearest-neighbor paths such that $x_0 = 0$ and $\tilde{x}_0 = x$. Then,

$$\begin{aligned}
&\big|\mathbb{E}\big[P^{\omega,\xi}\{\tilde{X}_{0,n} = \tilde{x}_{0,n}, X_{0,n} = x_{0,n}\}\big] - \mathbb{E}\big[P^{\omega,\xi}\{\tilde{X}_{0,n} = \tilde{x}_{0,n}\}\big]\mathbb{E}\big[P^{\omega,\xi}\{X_{0,n} = x_{0,n}\}\big]\big| \\
&\quad \leq \Psi(|x| - 2n) \\
&\quad \leq a_x = \begin{cases} \Psi\big(\frac{|x|}{2}\big) & \text{if } n < |x|/4 \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}$$

We can now apply the coupling Lemma 2.1 of [6] to get $P_{x,x}^\xi\{\bar{Y}_1 \neq \bar{\bar{Y}}_1\} \leq a_x$. Averaging out the coin ξ, we see that

$$\begin{aligned}
P_{x,x}\{\bar{Y}_1 \neq \bar{\bar{Y}}_1\} &\leq \Psi\left(\frac{|x|}{2}\right) + \mathbf{P}\left\{\tau_1 > \frac{|x|}{4L}\right\} \\
&\leq \Psi\left(\frac{|x|}{2}\right) + \big(1-\kappa\big)^{|x|/(4L)} \\
&\leq Cb^{-|x|}
\end{aligned}$$

for L large enough. $\square$

Now we will closely follow Appendix A in [44] of Rassoul-Agha and Seppäläinen, but adapt it to our conditions. Specifically, using the result of Lemma 34, the assumption $P_{x,x}\{\bar{Y}_1 \neq \bar{\bar{Y}}_1\} \leq C|x|^{-p}$ is replaced by $P_{x,x}\{\bar{Y}_1 \neq \bar{\bar{Y}}_1\} \leq Cb^{-|x|}$. Let $\mathbb{S}$ be a subgroup of $\mathbb{Z}^d$. Let $\bar{Y} = (\bar{Y}_k)_{k\geq 0}$ denote a Markov chain on $\mathbb{S}$ with transition probabilities $\bar{q}(x,y)$. Let $\bar{\bar{Y}} = (\bar{\bar{Y}}_k)_{k\geq 0}$ be a symmetric random walk on $\mathbb{S}$ with transition probabilities $\bar{\bar{q}}(x,y) = \bar{\bar{q}}(0, y-x) = \bar{\bar{q}}(y,x)$. Let y^i represent the i^{th} coordinate of the vector y. Denote $B_\rho = [-\rho,\rho]^d$, the cube with side of length ρ.

We will use the following properties:

Property 35 *The random walk is symmetric and has a finite moments:*

$$E_0\left[|\bar{\bar{Y}}_1|^m\right] \leq CL^m. \tag{3.13}$$

Moreover,

$$E_0\left[|\bar{\bar{Y}}_1|^2\right] \geq 2L.$$

Property 36 *The random walk $\bar{\bar{Y}}$ satisfies the following ellipticity condition for large L:*

$$P_0\big\{\bar{\bar{Y}}^j_1 \geq 1\big\} \geq \frac{1}{4}. \tag{3.14}$$

Property 37 *The Markov chain $\bar{Y}$ satisfies a uniform ellipticity condition: For any $\delta > 0$ and $1/2 < \gamma \leq 1$, there exists a constant C such that, for any $L > C$, x satisfying $x \geq \delta L$, and all j,*

$$P_x\big\{\bar{Y}^j_1 \geq x + L^{1-\gamma}\big\} \geq \frac{1-(1-\kappa)^{\delta/4}}{4}.$$

Property 38 *For $x \neq 0$, $\bar{q}$ and $\bar{\bar{q}}$ can be coupled so that $P_{x,x}\{\bar{Y}_1 \neq \bar{\bar{Y}}_1\} \leq Cb^{-|x|}$ where $0 < C < \infty$ is a constant independent of x and L, and $b = (1-\kappa)^{-1/(4L)}$.*

Property 39 *Abbreviate $\sigma^2 = E_0[|\bar{\bar{Y}}^j_1|^2]$.*

$$P_0\big\{\bar{\bar{Y}}^j_{n^2} < -n\big\} = \frac{1}{2}P_0\big\{|\bar{\bar{Y}}^j_{n^2}| > n\big\} \geq \frac{(\sigma^2-1)^2}{2\big(E\big[|\bar{\bar{Y}}^j_1|^4\big] + \sigma^4\big)} \geq \frac{1}{CL^2} \tag{3.15}$$

for a constant C, L large, and all n.

Property 40 *There exists a constant $C > 0$ such that for L large enough and all n,*

$$P_0\bigg\{\min_{i\leq n^2} \bar{\bar{Y}}^j_i \geq -n\bigg\} \geq P_0\big\{|\bar{\bar{Y}}^j_{n^2}| \leq n\big\} \geq \frac{1}{CL^2}. \tag{3.16}$$

Proof of Property 35:

To calculate the upper bound on the moments, we use that $|\bar{\bar{Y}}_1| \leq \tau_1 L$ to see that

$$E_0\big|\bar{\bar{Y}}_1\big|^m \leq \mathbf{E}\big[\tau_1^m\big]L^m \leq \kappa^{-m}L^m.$$

The lower bound on the second moment comes from calculating

$$\begin{aligned}
E_0\big[|\bar{\bar{Y}}_1|^2\big] &= E_{0,0}\big[E_{0,0}\big[\big|\tilde{X}_{\tau_1 L} - X_{\tau_1 L}\big|^2 \,\big|\, \tilde{X}_{\tau_1 L-L}, X_{\tau_1 L-L}\big]\big] \\
&\quad - \big|E_{0,0}\big[E_{0,0}\big[\tilde{X}_{\tau_1 L} - X_{\tau_1 L} \,\big|\, \tilde{X}_{\tau_1 L-L}, X_{\tau_1 L-L}\big]\big]\big|^2 \\
&\geq E_{0,0}\big[E_{0,0}\big[|\tilde{X}_{\tau_1 L} - X_{\tau_1 L}|^2 \,\big|\, \tilde{X}_{\tau_1 L-L}, X_{\tau_1 L-L}\big]\big] \\
&\quad - E_{0,0}\big[\big|E_{0,0}\big[\tilde{X}_{\tau_1 L} - X_{\tau_1 L} \,\big|\, \tilde{X}_{\tau_1 L-L}, X_{\tau_1 L-L}\big]\big|^2\big] \\
&= E_{0,0}\big[\mathrm{Var}\big(\tilde{X}_{\tau_1 L} - X_{\tau_1 L} \,\big|\, \tilde{X}_{\tau_1 L-L}, X_{\tau_1 L-L}\big)\big] \\
&= 2L.
\end{aligned}$$

The last equality is due to the fact that, between levels $\tau_1 L - L$ and $\tau_1 L$, both walks perform a simple symmetric random walk. $\square$

Proof of Property 36:

To see the ellipticity of the random walk $\bar{\bar{Y}}$, first note that $P_0\big\{\bar{\bar{Y}}_1^j \geq 1\big\} = \frac{1}{2}P_0\big\{\bar{\bar{Y}}_1^j \neq 0\big\}$ by symmetry. Now we will calculate an upper bound on $P_0\{\bar{\bar{Y}}_1^j = 0\}$:

$$\begin{aligned}
P_0\big\{\bar{\bar{Y}}_1^j = 0\big\} &= E_{0,0}\big[P_{0,0}\big\{\tilde{X}_{\tau_1 L}^j - X_{\tau_1 L}^j = 0 \,\big|\, X_{\tau_1 L-L}, \tilde{X}_{\tau_1 L-L}\big\}\big] \\
&\leq \max_{x\in\mathbb{Z}^d} P_x\{Z_L = 0\} \leq P_0\{Z_L = 0\} \leq L^{-1/2},
\end{aligned}$$

where Z represents a symmetric random walk with steps of -2 and 2 with probability 1/4 each, and a step of 0 with probability 1/2. Rearranging this, we get that $P_0\{\bar{\bar{Y}}_1^j \neq 0\} \geq 1 - L^{-1/2} \geq \frac{1}{2}$ for large L. $\square$

Proof of Property 37:

To show the ellipticity of the Markov chain, we will bound $1 - P_x\{|\bar{Y}_1^j| \geq x + L^{1-\gamma}\}$ from above:

$$\begin{aligned}
P_x\big\{|\bar{Y}_1^j| \leq x + L^{1-\gamma}\big\} &\leq P_x\big\{\bar{Y}_1^j \neq \bar{\bar{Y}}_1^j\big\} + P_x\big\{|\bar{\bar{Y}}_1^j| \leq x + L^{1-\gamma}\big\} \\
&\leq (1-\kappa)^{\delta/4} + P_0\big\{|\bar{\bar{Y}}_1^j| \leq L^{1-\gamma}\big\}.
\end{aligned}$$

To bound the second term, let Z represent a symmetric random walk with steps -2 and 2 with probability 1/4 each, and a step of 0 with probability 1/2. Then,

$$P_0\big\{|\bar{\bar{Y}}_1^j| \leq L^{1-\gamma}\big\} = E_{0,0}\big[E_{0,0}\big[\mathbb{1}\big\{|\tilde{X}_{\tau_1 L} - X_{\tau_1 L}| \leq L^{1-\gamma}\big\} \,\big|\, \tilde{X}_{\tau_1 L-L}, X_{\tau_1 L-L}\big]\big]$$

$$
\begin{aligned}
&\leq \max_x P_x\big\{|Z_L| \leq L^{1-\gamma}\big\}\\
&= P_0\big\{|Z_L| \leq L^{1-\gamma}\big\}\\
&\leq CL^{1/2-\gamma},
\end{aligned}
$$

where the last inequality is by P6 on page 72 of [49]. For $\gamma > 1/2$ and L large enough, $CL^{1/2-\gamma} \leq (1-(1-\kappa)^{\delta/4})/2$. □

Proof of Property 39:

To see that this property holds, first write

$$
\begin{aligned}
n^2\sigma^2 = E_0\big[|\bar{\bar{Y}}^j_{n^2}|^2\big] &= E_0\big[|\bar{\bar{Y}}^j_{n^2}|^2 \cdot \mathbb{1}\big\{|\bar{\bar{Y}}^j_{n^2}| \leq n\big\}\big] + E_0\big[|\bar{\bar{Y}}^j_{n^2}|^2 \cdot \mathbb{1}\big\{|\bar{\bar{Y}}^j_{n^2}| > n\big\}\big]\\
&\leq n^2 + E_0\big[|\bar{\bar{Y}}^j_{n^2}|^4\big]^{1/2} P_0\big\{|\bar{\bar{Y}}^j_{n^2}| > n\big\}^{1/2}.
\end{aligned}
$$

Rearranging this and using Property 35, we have (3.15). □

Proof of Property 40:

The first inequality in (3.16) comes from the reflection principle by writing

$$
\begin{aligned}
P_0\big\{\bar{\bar{Y}}^j_{n^2} < -n\big\} &= \sum_{k=1}^{n^2} P_0\big\{\bar{\bar{Y}}^j_i \geq -n, i \leq k-1, \bar{\bar{Y}}^j_k < -n, \bar{\bar{Y}}^j_{n^2} < -n\big\}\\
&\geq \frac{1}{2}\sum_{k=1}^{n^2} P_0\big\{\bar{\bar{Y}}^j_i \geq -n, i \leq k-1, \bar{\bar{Y}}^j_k < -n\big\}\\
&= \frac{1}{2} P_0\left\{\min_{i \leq n^2} \bar{\bar{Y}}^j_i \geq -n\right\}
\end{aligned}
$$

and using $2P\{\bar{\bar{Y}}^j_{n^2} < -n\} = P\{|\bar{\bar{Y}}^j_{n^2}| < n\}$. The second inequality in (3.16) uses Chebyshev's exponential inequality and Taylor's expansion. To see this, take $a \in (0, -\log\kappa)$ small enough so that $aE[\tau_1^2 e^{a\tau_1}] < 1/2$ and write

$$
\begin{aligned}
P_0\left\{\frac{\bar{\bar{Y}}^j_{n^2}}{n} \geq 1\right\} &\leq e^{-a/L^2} E_0\big[e^{a\bar{\bar{Y}}^j_{n^2}/(L^2 n)}\big] = e^{-a/L^2} E_0\big[e^{a\bar{\bar{Y}}^j_1/(L^2 n)}\big]^{n^2}\\
&\leq e^{-a/L^2}\left(1 + \frac{a^2}{2L^4 n^2} E_0\big[|\bar{\bar{Y}}^j_1|^2 e^{a\bar{\bar{Y}}^j_1/(L^2 n)}\big]\right)^{n^2}\\
&\leq e^{-a/L^2}\left(1 + \frac{a^2}{2L^4 n^2}\mathbf{E}\big[L^2\tau_1^2 e^{a\tau_1 L/(L^2 n)}\big]\right)^{n^2}\\
&\leq e^{-a/L^2}\left(1 + \frac{a^2}{2L^2 n^2}\mathbf{E}\big[\tau_1^2 e^{a\tau_1}\big]\right)^{n^2}\\
&\leq e^{-1/(2L^2)} \leq 1 - \frac{1}{CL^2}
\end{aligned}
$$

for L large enough. □

In the rest of this section we show that if $L = \beta \log n$ with $\beta > 0$, and $r = n^{\varepsilon}$ with $\varepsilon > 0$ small,

$$\sum_{k=0}^{n-1} P_z\{\bar{Y}_k \in B_{Lr}\} \leq Cn^{1-\eta} \tag{3.17}$$

for some $0 < \eta < 1$. Note that line (3.17) is larger for $z \in B_{Lr}$ than for $z \notin B_{Lr}$, so we will assume that $z \in B_{Lr}$.

The proof of (3.17) will be shown in two phases. We will first show an exit time bound, then we will demonstrate that the Markov chain $\bar{Y}$ follows the random walk $\bar{\bar{Y}}$ in excursions outside B_{Lr} often enough. Since the excursions for the random walk are long, $\bar{Y}$ will spend little enough time in B_{Lr}, and the upper bound (3.17) will be achieved.

In the following Lemmas (41–48), L is a fixed integer. We will, however, eventually let L depend on n. Therefore, we will track down any dependence on L in our estimates.

Lemma 41 *Let $\zeta = \inf\{n \geq 1 : \bar{\bar{Y}}_n \in A\}$ be the random walk $\bar{\bar{Y}}$'s first entrance time into some set $A \subseteq \mathbb{S}$. We can then couple the Markov Chain $\bar{Y}$ and the random walk $\bar{\bar{Y}}$ such that*

$$P_{x,x}\{\bar{Y}_k \neq \bar{\bar{Y}}_k \text{ for some } k \leq \zeta\} \leq CE_x \sum_{k=0}^{\zeta-1} b^{-|\bar{\bar{Y}}_k|},$$

where C is a constant independent of L.

Proof:

For each state x, create an i.i.d. sequence $(\bar{Z}_k^x, \bar{\bar{Z}}_k^x)_{k\geq 1}$ such that $\bar{Z}_k^x$ has distribution $\bar{q}(x, x+\cdot)$ and $\bar{\bar{Z}}_k^x$ has distribution $\bar{\bar{q}}(x, x+\cdot) = \bar{\bar{q}}(0, \cdot)$. Also, each pair $(\bar{Z}_k^x, \bar{\bar{Z}}_k^x)$ is coupled such that $P\{\bar{Z}_k^x \neq \bar{\bar{Z}}_k^x\} \leq Cb^{-|x|}$, and for distinct x these sequences are independent.

We will construct the process $(\bar{Y}_n, \bar{\bar{Y}}_n)$ as follows: Define

$$\bar{L}_n(x) = \sum_{k=0}^{n} \mathbb{1}\{\bar{Y}_k = x\} \text{ and } \bar{\bar{L}}_n(x) = \sum_{k=0}^{n} \mathbb{1}\{\bar{\bar{Y}}_k = x\} \text{ for } n \geq 0.$$

Given the initial point $(\bar{Y}_0, \bar{\bar{Y}}_0)$, define for $n \geq 1$

$$\bar{Y}_n = \bar{Y}_{n-1} + \bar{Z}^{\bar{Y}_{n-1}}_{\bar{L}_{n-1}(\bar{Y}_{n-1})} \text{ and } \bar{\bar{Y}}_n = \bar{\bar{Y}}_{n-1} + \bar{\bar{Z}}^{\bar{\bar{Y}}_{n-1}}_{\bar{\bar{L}}_{n-1}(\bar{\bar{Y}}_{n-1})}.$$

By this construction of the coupling, if $\bar{Y}_k = \bar{\bar{Y}}_k$ for $0 \leq k < n$ and $\bar{Y}_n = \bar{\bar{Y}}_n = x$, the probability that $\bar{Y}_{n+1} \neq \bar{\bar{Y}}_{n+1}$ is bounded by $Cb^{-|x|}$. Then:

$$P_{x,x}\{\bar{Y}_k \neq \bar{\bar{Y}}_k \text{ for some } 1 \leq k \leq \zeta\}$$

$$\le \sum_{k=1}^{\infty} P_{x,x}\big\{\bar{Y}_j = \bar{\bar{Y}}_j \in A^c \text{ for } 1 \le j < k, \bar{Y}_k \neq \bar{\bar{Y}}_k\big\}$$
$$\le \sum_{k=1}^{\infty} E_{x,x}\big[\mathbb{1}\big\{\bar{Y}_j = \bar{\bar{Y}}_j \in A^c \text{ for } 1 \le j < k\big\} P_{\bar{Y}_{k-1},\bar{\bar{Y}}_{k-1}}\big\{\bar{Y}_1 \neq \bar{\bar{Y}}_1\big\}\big]$$
$$\le C\sum_{k=1}^{\infty} E_{x,x}\big[\mathbb{1}\big\{\bar{Y}_j = \bar{\bar{Y}}_j \in A^c \text{ for } 1 \le j < k\big\} b^{-|\bar{\bar{Y}}_{k-1}|}\big]$$
$$\le CE_x \sum_{m=0}^{\zeta-1} b^{-|\bar{\bar{Y}}_m|},$$

as claimed. $\square$

Lemma 42 *Fix a coordinate index $j \in \{1,\dots,d\}$. Let r_0 be a positive integer and $\bar{\bar{w}} = \inf\{n \ge 1 : \bar{\bar{Y}}_n^j \le r_0\}$ be the first time $\bar{\bar{Y}}$ enters the half-space $\mathcal{H} = \{x : x^j \le r_0\}$. Couple $\bar{Y}$ and $\bar{\bar{Y}}$ starting from some initial point $x \in \mathcal{H}^c$. Then there is a constant C independent of r_0 and L such that*

$$\sup_{x\in\mathcal{H}^c} P_{x,x}\big\{\bar{Y}_k \neq \bar{\bar{Y}}_k \textit{ for some } k \in \{1,\dots,\bar{\bar{w}}\}\big\} \le \frac{CL^{12}}{(b-1)^2}\, b^{-r_0} \textit{ for all } r_0 \ge 1.$$

Likewise, the result also holds for $\mathcal{H} = \{x : x^j \ge -r_0\}$.

Proof:
By the previous lemma,

$$\begin{aligned} P_{x,x}\big\{\bar{Y}_k \neq \bar{\bar{Y}}_k \text{ for some } k \in \{1,\dots,\bar{\bar{w}}\}\big\} &\le CE_x \sum_{k=0}^{\bar{\bar{w}}-1} b^{-|\bar{\bar{Y}}_k|} \\ &\le CE_{x^j} \sum_{k=0}^{\bar{\bar{w}}-1} b^{-|\bar{\bar{Y}}_k^j|} \\ &= C\sum_{t=r_0+1}^{\infty} b^{-t} g(x^j,t), \end{aligned}$$

where for $s,t \in [r_0+1,\infty)$,

$$g(s,t) = \sum_{n=0}^{\infty} P_s\big\{\bar{\bar{Y}}_n^j = t, \bar{\bar{w}} > n\big\}. \tag{3.18}$$

Then, following Sections 18 and 19 in [49], we get the bound

$$g(s,t) \le C\big(1 + (s-r_0-1)\wedge(t-r_0-1)\big) \le c^{-1}(t-r_0),\ \ s,t \in [r_0+1,\infty). \tag{3.19}$$

See the paragraph after (A.5) in [44] for the details. The constant c^{-1} in (3.19) is bounded by CL^{12}. Indeed, from D2 on page 201 of [49], the constant c^{-1} satisfies

$$c^{-1} = e^{\sum_{k=1}^{\infty} k^{-1} P_0(\bar{\bar{Y}}_k^j = 0)}, \tag{3.20}$$

and by Theorem 1 on page 612 of [26],

$$c^{-1} = 2\,\frac{E\big[\bar{\bar{Y}}_N^j\big]^2}{E\big[|\bar{\bar{Y}}_1^j|^2\big]},$$

where N is the first time $\bar{\bar{Y}}_n^j > 0$. By Lemma 5.1.10 of [36], as well as our lines (3.13), (3.14), and (3.15),

$$c^{-1} \leq C\frac{\big(E\big[|\bar{\bar{Y}}_1^j|^4\big] + \sigma^4\big)^2 E_0\big[|Y_1^j|^2\big]^2}{L^4\big(\sigma^2 - 1\big)^4}\,\frac{E_0\big[|Y_1^j|^2\big]}{E_0\big[|Y_1^j|^2\big]} \leq CL^{12}.$$

Using this, we get that

$$E_x \sum_{k=0}^{\bar{\bar{w}}-1} b^{-|\bar{\bar{Y}}_k|} \leq c^{-1} \sum_{t>r_0}^{\infty} (t - r_0) b^{-t} \leq c^{-1} b^{-r_0} \sum_{t>0}^{\infty} t b^{-t} \leq \frac{CL^{12}}{(b-1)^2}\, b^{-r_0}.$$

Since this upper bound does not depend on x, we are done. □

Lemma 43 *For L large and for any positive integers $r_0 < r$ that satisfy*

$$r(1-\kappa)^{-r_0/4} < L^{-37}, \tag{3.21}$$

the following holds:

$$\inf_{x \in B_{Lr} \backslash B_{Lr_0}} P_x\big\{\textit{without entering } B_{Lr_0} \textit{ chain } \bar{Y} \textit{ exits } B_{Lr} \textit{ by time } L^{27} r^3\big\} \geq \frac{1}{L^{23} r}.$$

Proof:

Let $x \in B_{Lr} \backslash B_{Lr_0}$. Then, x has a coordinate $x^j \in [-Lr, -Lr_0 - 1] \cup [Lr_0 + 1, Lr]$. We will assume that $x^j \in [Lr_0 + 1, Lr]$, since the same argument works for $x^j \in [-Lr, -Lr_0 - 1]$.

We will analyze the above event by considering the walk given by the j^{th} coordinate of $\bar{\bar{Y}}$. The event in question happens if, starting at x^j, $\bar{\bar{Y}}^j$ exits $[Lr_0 + 1, Lr]$ by time $L^{27} r^3$ into the interval $[Lr + 1, \infty)$, and $\bar{Y}$ and $\bar{\bar{Y}}$ stay coupled together for this time. Let $\bar{\bar{\zeta}}$ be the time $\bar{\bar{Y}}^j$ exits $[Lr_0 + 1, Lr]$ and $\bar{\bar{w}}$ be the time $\bar{\bar{Y}}^j$ enters $(-\infty, Lr_0]$, so $\bar{\bar{w}} \geq \bar{\bar{\zeta}}$. Then, the complementary probability is bounded above by

$$P_{x^j}\big\{\bar{\bar{Y}}^j \text{ exits } [Lr_0 + 1, Lr] \text{ into } (-\infty, Lr_0]\big\} + P_{x^j}\big\{\bar{\bar{\zeta}} > L^{27} r^3\big\}$$

$$+ P_{x,x}\{\bar{Y}_k \neq \bar{\bar{Y}}_k \text{ for some } k \in \{1, \ldots, \bar{\bar{w}}\}\}. \quad (3.22)$$

We will consider the terms one at a time. Let us now consider the first term. By pages 253–255 of [49], we get that

$$P_{x^j}\{\bar{\bar{Y}}^j \text{ exits } [Lr_0+1, Lr] \text{ into } [Lr+1,\infty)\} \geq \frac{x^j - Lr_0 - 1 - c_1}{Lr - Lr_0 - 1}. \quad (3.23)$$

See the paragraph after (A.9) of [44] for the details. Here,

$$\begin{aligned} c_1 &= c^{-1}\sum_{s=0}^{\infty}(1+s)a(s) \leq 2c^{-1}\sum_{s=0}^{\infty} sa(s) = 2c^{-1}\sum_{k=1}^{\infty}\sum_{s=0}^{\infty} skP_0\{\bar{\bar{Y}}_1^j = s+k\} \\ &= 2c^{-1}\sum_{\ell=1}^{\infty}\sum_{s=0}^{\ell-1} s(\ell-s)P_0\{\bar{\bar{Y}}_1^j = \ell\} \leq 2c^{-1}E_0\big[|\bar{\bar{Y}}_1|^3\big] \leq CL^{15}, \end{aligned}$$

where c^{-1} is as defined in line (3.20) and $a(s)$ is the potential kernel defined by the second equality.

This probability is minimized when $x^j = Lr_0 + 1$. From this x^j, there exists a fixed positive probability α_2 to overtake $c_1 + Lr_0 + 2$ before it goes below Lr_0. Then, after the walk passes $c_1 + 1 + Lr_0$, use (3.23) to get

$$P_{x^j}\{\bar{\bar{Y}}^j \text{ exits } [Lr_0+1, Lr] \text{ into } [Lr+1,\infty)\} \geq \frac{\alpha_2}{Lr - Lr_0 - 1} \geq \frac{\alpha_2}{Lr}.$$

As a result, we see that

$$P_{x^j}\{\bar{\bar{Y}}^j \text{ exits } [Lr_0+1, Lr] \text{ into } (-\infty, Lr_0]\} \leq 1 - \frac{\alpha_2}{Lr} \quad (3.24)$$

uniformly over $Lr_0 < x^j \leq Lr$.

To get a lower bound on α_2, we need the following Gambler's Ruin estimate:

Lemma 44 *For L large enough:*

$$P_1\{\bar{\bar{Y}}^j \textit{ exits } [1,a) \textit{ into } [a,\infty)\} \geq \frac{1}{CL^7 a}.$$

We will show the proof after the current lemma is finished. Lemma 44 shows that

$$\alpha_2 \geq \frac{1}{CL^{22}}.$$

Let us now bound the second term in (3.22). Let $g(s,t)$ be the Green function of the random walk $\bar{\bar{Y}}^j$ for the half-line $(-\infty, Lr_0]$ as in (3.19), and let $\tilde{g}(s,t)$ be the Green function for the complement of the interval $[Lr_0+1, Lr]$. Then, $\tilde{g}(s,t) \le g(s,t)$, and we get the moment bound by (5.5) on page 108 of [36]

$$E_{x^j}\left[\bar{\bar{\zeta}}\right] = \sum_{t=Lr_0+1}^{Lr} \tilde{g}(x^j,t) \le \sum_{t=Lr_0+1}^{Lr} g(x^j,t) \le CL^4 r^2.$$

Therefore, by Chebyshev's inequality,

$$P_{x^j}\big\{\bar{\bar{\zeta}} > L^{27} r^3\big\} \le \frac{C}{L^{23} r} \le \frac{\alpha_2}{4r} \tag{3.25}$$

for L large, uniformly over $x^j \in [Lr_0+1, Lr]$.

By Lemma 42 and (3.21), the last probability in (3.22) is bounded above by

$$\frac{CL^{12}}{(b-1)^2} b^{-Lr_0} \le CL^{14}(1-\kappa)^{r_0/4} \le \frac{\alpha_2}{4r}$$

for large L. By combining this with lines (3.24) and (3.25), we see that (3.22) is bounded above by $1 - \frac{\alpha_2}{2r} \le 1 - \frac{1}{L^{23} r}$. As a result,

$$P_x\big\{\text{without entering } B_{Lr_0} \text{ chain } \bar{Y} \text{ exits } B_{Lr} \text{ by time } A_1 r^3\big\} \ge \frac{1}{L^{23} r}$$

for all $x \in B_{Lr} \backslash B_{Lr_0}$. □

Proof of Lemma 44:

We follow Section 5.1.1 of [36] closely. We want a lower bound on

$$p_a = P_1\big\{\bar{\bar{Y}}^j \text{ exits } [1,a) \text{ into } [a,\infty)\big\}.$$

Now, by our (3.16) and (3.15), and by (5.7) on page 110 of [36], we have the upper bound

$$\sum_{k=1}^{a} g(1,k) \le CL^5 a,$$

where g was defined in (3.18) with $r_0 = 0$.

Let $T_a = \min\{n > 0 : \bar{\bar{Y}}^j_n \notin [1,a)\}$ and $N = \min\{n > 0 : \bar{\bar{Y}}^j_N \le 0\}$. We have

$$\begin{aligned}
P_1\big\{|\bar{\bar{Y}}^j_{T_a}| \ge s+a\big\} &= \sum_{\ell=0}^{\infty} P_1\big\{T_a = \ell+1, |\bar{\bar{Y}}^j_{T_a}| \ge s+a\big\} \\
&\le \sum_{\ell=0}^{\infty} P_1\big\{T_a > \ell, |\bar{\bar{Y}}^j_{T_a} - \bar{\bar{Y}}^j_{T_a-1}| \ge s\big\} \\
&\le P_0\big\{|\bar{\bar{Y}}^j_1| \ge s\big\} \sum_{\ell=0}^{\infty} P_1\big\{T_a > \ell\big\}
\end{aligned}$$

$$\le P_0\big\{|\bar{\bar{Y}}_1^j| \ge s\big\} \sum_{\ell=0}^{\infty} P_1\big\{N > \ell, \bar{\bar{Y}}_\ell^j \le a\big\}$$
$$= P_0\big\{|\bar{\bar{Y}}_1^j| \ge s\big\} \sum_{k=1}^{a} g(1,k)$$
$$\le 2L^5 a P_0\big\{|\bar{\bar{Y}}_1^j| \ge s\big\}.$$

If $t > 0$,

$$\begin{aligned} E_1\big[|\bar{\bar{Y}}_{T_a}^j| \cdot \mathbb{1}\{|\bar{\bar{Y}}_{T_a}^j| \ge (1+t)a\}\big] &= \int_{ta}^{\infty} P_1\big\{|\bar{\bar{Y}}_{T_a}^j| \ge s + a\big\}\, ds \\ &\le CL^5 a \int_{ta}^{\infty} P_0\big\{|\bar{\bar{Y}}_1^j| \ge s\big\}\, ds \\ &= CL^5 E_0\big[|\bar{\bar{Y}}_1^j| \cdot \mathbb{1}\{|\bar{\bar{Y}}_1^j| \ge ta\}\big] \\ &\le \frac{CL^5}{t} E_0\big[|\bar{\bar{Y}}_1^j|^2\big] \le \frac{CL^7}{t}. \end{aligned}$$

Choosing $t = 2CL^7$, we see that

$$E_1\big[|\bar{\bar{Y}}_{T_a}^j| \cdot \mathbb{1}\{|\bar{\bar{Y}}_{T_a}^j| \ge (1+t)a\}\big] \le \frac{1}{2}.$$

Consider the martingale $M_k = \bar{\bar{Y}}_{k \wedge T_a}^j$. By the optional stopping theorem,

$$1 = E_1[M_0] = E_1[M_\infty] \le E_1\big[\bar{\bar{Y}}_{T_a}^j \cdot \mathbb{1}\big\{\bar{\bar{Y}}_{T_a}^j \ge a\big\}\big].$$

Therefore,

$$a(1+t)p_a \ge E_1\big[\bar{\bar{Y}}_{T_a}^j \cdot \mathbb{1}\big\{a \le \bar{\bar{Y}}_{T_a}^j \le (1+t)a\big\}\big] \ge \frac{1}{2}.$$

Now, for L large we get the bound

$$p_a \ge \frac{1}{CL^7 a}.$$

The lemma is proved. □

Lemma 45 *Consider positive integers r_0 and r that satisfy*

$$\log\log r \le r_0 \le 2\log\log r < r.$$

There exists a constant $C > 0$ such that for $L > C$ and $r > C$,

$$\inf_{x \in B_{Lr} \setminus B_{Lr_0}} P_x\big\{\textit{without entering } B_{Lr_0} \textit{ chain } \bar{Y} \textit{ exits } B_{Lr} \textit{ by time } L^{27} r^4\big\}$$
$$\ge (\log r)^{-46\log L} r^{-3}.$$

Proof:

Let $r_k = r_0^{3^k}$ for $k \geq 0$ and $t_n = L^{27} \sum_{k=1}^{n} r_k^3$.

We will begin by showing that for $n \geq 1$,

$$\inf_{x \in B_{Lr_n} \backslash B_{Lr_0}} P_x\big\{\text{without entering } B_{Lr_0} \text{ chain } \bar{Y} \text{ exits } B_{Lr_n} \text{ by time } t_n\big\}$$
$$\geq \prod_{k=1}^{n} \left(\frac{1}{L^{23} r_k}\right). \tag{3.26}$$

We will prove (3.26) by induction. The case $n = 1$ is Lemma 43 using $r_1 = r_0^3$ and r_0. r_0 needs to be taken large enough such that $\rho^3(1-\kappa)^{\rho/4}$ is decreasing for $\rho \geq r_0$. Now, assume that (3.26) holds for n and consider exiting $B_{Lr_{n+1}}$ without entering B_{Lr_0}. If the initial state x is in $B_{Lr_n} \backslash B_{Lr_0}$, by induction we know that, with probability bounded below by $\prod_{k=1}^{n}(1/(L^{23} r_k))$, the chain first takes time t_n to exit B_{Lr_n} without entering B_{Lr_0}. If the walk landed in $B_{Lr_{n+1}} \backslash B_{Lr_0}$, take another time $L^{27} r_{n+1}^3 = L^{27} r_0^{3^{n+2}}$ to exit $B_{Lr_{n+1}}$ without entering B_{Lr_n} with probability at least $1/(L^{23} r_{n+1})$ by Lemma 43. Then, the times taken add to t_{n+1} and the probabilities multiply to $\prod_{k=1}^{n+1} 1/(L^{23} r_k)$.

If the initial state x lies in $B_{Lr_{n+1}} \backslash B_{Lr_n}$, then we can apply Lemma 43 to see that $\bar{Y}$ exits $B_{Lr_{n+1}}$ without entering B_{Lr_n} in time $L^{27} r_{n+1}^3 = L^{27} r_0^{3^{n+2}}$ with probability at least $1/(L^{23} r_{n+1})$. This completes the proof of (3.26).

Now, let $N = \min\{k \geq 1 : r_k \geq r\}$, so $r_0^{3^{N-1}} < r$. If $\log\log r > e^e$, then $\log\log r_0 > 0$ and $N < 1 + (\log\log r)/(\log 3) < 2\log\log r$. First, we will take $n = N-1$ in (3.26). This allows $\bar{Y}$ to exit B_{Lr} without entering $B_{Lr_{N-1}}$. The probability of achieving this is bounded below by

$$\prod_{k=1}^{N-1} \left(\frac{1}{L^{23} r_k}\right) \cdot \frac{1}{L^{23} r} \geq \frac{1}{L^{23N}} r_0^{-3^N/2} r^{-1} \geq (\log r)^{-46 \log L} r^{-3/2-1} \geq (\log r)^{-46 \log L} r^{-3}.$$

Also, we get the bound

$$t_{N-1} + L^{27} r^3 \leq L^{27}(N-1) r_0^{3^N} + L^{27} r^3 \leq L^{27} r^4$$

for the elapsed time. $\square$

Lemma 46 *Let $U = \inf\{n \geq 0 : \bar{Y}_n \notin B_{Lr}\}$ be the first exit time from B_{Lr} for the Markov chain $\bar{Y}$. There exists a constant C such that if $L > C$ and $r > C$ are positive integers satisfying*

$$(\log r)^{46 \log L} < r \quad \textit{and} \quad \left(\frac{2d}{\kappa}\right)^{\delta L} + \left(\frac{1-(1-\kappa)^{\delta/4}}{4}\right)^{2L^\gamma \log\log r} < r$$

for some $\delta > 0$ and $1/2 < \gamma \leq 1$, then $\sup_{x \in B_{Lr}} E_x[U] \leq L^{28} r^{16}$.

Proof:

We know that $\sup_{x\in B_{Lr}} E_x[U] < \infty$ by ellipticity. Let $r_0 < r$ be positive integers with $\log\log r \le r_0 \le 2\log\log r$, and let L and r be large enough for the conditions of Lemma 45 to be met.

Let $0 = T_0 = S_0 \le T_1 \le S_1 \le T_2 \le \cdots$ be the successive exit and entrance times into B_{Lr_0}. Precisely, for $i \ge 1$ while $S_{i-1} < \infty$,

$$T_i = \inf\{n \ge S_{i-1} : \bar{Y}_n \notin B_{Lr_0}\} \quad \text{and} \quad S_i = \inf\{n \ge T_i : \bar{Y}_n \in B_{Lr_0}\}.$$

Then, if $S_i = \infty$ for some i, we set $T_j = S_j = \infty$ for all $j > i$. Also, if $\bar{Y}_0 \in B_{Lr}\backslash B_{Lr_0}$, then $T_1 = 0$. For $x \in B_{Lr_0}$, ellipticity implies that the expected value of the exit time from $B_{\delta L}$ is bounded by $(2d/\kappa)^{\delta L}$. Then, Property 37 implies that

$$\sup_{x\in B_{Lr_0}} E_x[T_1] \le \left(\frac{2d}{\kappa}\right)^{\delta L} + \left(\frac{1-(1-\kappa)^{\delta/4}}{4}\right)^{2L^\gamma \log\log r} < r, \tag{3.27}$$

so we see that T_1 is finite, but $S_1 = \infty$ is allowed, as it is possible that $\bar{Y}$ never returns to B_{Lr_0} after it leaves. Since $T_1 \le U < \infty$, we can calculate $E_x[U]$ as follows for $x \in B_{Lr}$.

$$\begin{aligned} E_x\big[U\big] &= \sum_{j=1}^{\infty} E_x\big[U,\ T_j \le U < S_j\big] \\ &= \sum_{j=1}^{\infty} E_x\big[T_j,\ T_j \le U < S_j\big] + \sum_{j=1}^{\infty} E_x\big[U - T_j,\ T_j \le U < S_j\big]. \end{aligned} \tag{3.28}$$

We will first focus on the last sum in (3.28). Using Lemma 45 inductively, for any $z \in B_{Lr}\backslash B_{Lr_0}$,

$$\begin{aligned} &P_z\big\{U > jL^{27}r^4,\ U < S_1\big\} \\ &\le P_z\big\{\bar{Y}_k \in B_{Lr}\backslash B_{Lr_0} \text{ for } k \le jL^{27}r^4\big\} \\ &= E_z\big[\mathbb{1}\big\{\bar{Y}_k \in B_{Lr}\backslash B_{Lr_0} \text{ for } k \le (j-1)L^{27}r^4\big\} P_{\bar{Y}_{(j-1)L^{27}r^4}}\big\{\bar{Y}_k \in B_{Lr}\backslash B_{Lr_0} \text{ for } k \le L^{27}r^4\big\}\big] \\ &\le \cdots \le \big(1 - (\log r)^{-46\log L} r^{-3}\big)^j \le (1 - r^{-4})^j. \end{aligned}$$

Using this, we see that for $z \in B_{Lr}\backslash B_{Lr_0}$,

$$\begin{aligned} E_z\big[U,\ U < S_1\big] &= \sum_{m=0}^{\infty} P_z\big\{U > m,\ U < S_1\big\} \\ &\le L^{27}r^4 \sum_{j=1}^{\infty} P_z\big\{U > jL^{27}r^4,\ U < S_1\big\} \end{aligned}$$

$$\leq L^{27} r^4 \sum_{j=1}^{\infty} \left(1 - r^{-4}\right)^j \leq L^{27} r^8. \tag{3.29}$$

Now, we will consider the failure to exit B_{Lr} during earlier excursions in $B_{Lr} \backslash B_{Lr_0}$. Let

$$H_i = \left\{\bar{Y}_n \in B_{Lr} \text{ for } T_i \leq n < S_i\right\}$$

be the event that the chain $\bar{Y}$ does not exit B_{Lr} between the i^{th} exit from and entrance back into B_{Lr_0}. As a result of Lemma 45, note that

$$P_x\left\{H_i \,\middle|\, \mathcal{F}_{T_i}\right\} \leq 1 - (\log r)^{-46 \log L} r^{-3} \leq 1 - r^4 \text{ for } i \geq 1, \text{ on the event } \left\{T_i < \infty\right\}. \tag{3.30}$$

Using this, we see that:

$$\begin{aligned}
E_x\left[U - T_j,\ T_j \leq U < S_j\right] &= E_x\left[\prod_{k=1}^{j-1} \mathbb{1}_{H_k} \cdot \mathbb{1}\left\{T_j < \infty\right\} \cdot E_{\bar{Y}_{T_j}}\left[U, U < S_1\right]\right] \\
&\leq L^{27} r^8 E_x\left[\prod_{k=1}^{j-1} \mathbb{1}_{H_k} \cdot \mathbb{1}\left\{T_{j-1} < \infty\right\}\right] \\
&\leq L^{27} r^8 \left(1 - r^{-4}\right)^{j-1}.
\end{aligned}$$

If $\bar{Y}_{T_j}$ lies outside B_{Lr}, the chain has already exited B_{Lr}, so $E_{\bar{Y}_{T_j}}(U) = 0$. In the other case, $\bar{Y}_{T_j} \in B_{Lr} \backslash B_{Lr_0}$ and (3.29) applies. Therefore, we can now bound the second sum in (3.28) by

$$\sum_{j=1}^{\infty} E_x\left[U - T_j,\ T_j \leq U < S_j\right] \leq L^{27} r^8 \sum_{j=1}^{\infty} \left(1 - r^{-4}\right)^{j-1} \leq L^{27} r^{12}.$$

We will now bound the first sum in (3.28). We will consider the $i = 0$ term separately, using (3.27) and (3.30) to see that

$$\begin{aligned}
E_x\left[T_j,\ T_j \leq U < S_j\right] &\leq \sum_{i=0}^{j-1} E_x\left[\prod_{k=1}^{j-1} \mathbb{1}_{H_k} \cdot \mathbb{1}\left\{T_j < \infty\right\} \cdot \left(T_{i+1} - T_i\right)\right] \\
&\leq r\left(1 - r^{-4}\right)^{j-1} \\
&\quad + \sum_{i=1}^{j-1} E_x\left[\prod_{k=1}^{i-1} \mathbb{1}_{H_k} \cdot \left(T_{i+1} - T_i\right) \cdot \mathbb{1}_{H_i} \cdot \mathbb{1}\left\{T_{i+1} < \infty\right\}\right] \\
&\quad \times \left(1 - r^{-4}\right)^{j-1-i}.
\end{aligned}$$

We will split the last expectation as:

$$E_x\left[\prod_{k=1}^{i-1} \mathbb{1}_{H_k} \cdot \left(T_{i+1} - T_i\right) \cdot \mathbb{1}_{H_i} \cdot \mathbb{1}\left\{T_{i+1} < \infty\right\}\right]$$

$$
\begin{aligned}
&\le E_x\Big[\prod_{k=1}^{i-1}\mathbb{1}_{H_k}\cdot\big(T_{i+1}-S_i\big)\cdot\mathbb{1}_{H_i}\cdot\mathbb{1}\big\{S_i<\infty\big\}\Big]\\
&\quad+E_x\Big[\prod_{k=1}^{i-1}\mathbb{1}_{H_k}\cdot\big(S_i-T_i\big)\cdot\mathbb{1}_{H_i}\cdot\mathbb{1}\big\{T_i<\infty\big\}\Big]\\
&\le E_x\Big[\prod_{k=1}^{i-1}\mathbb{1}_{H_k}\cdot\mathbb{1}\big\{S_i<\infty\big\}\cdot E_{\bar{Y}_{S_i}}\big[T_1\big]\Big]+E_x\Big[\prod_{k=1}^{i-1}\mathbb{1}_{H_k}\cdot\mathbb{1}\big\{T_i<\infty\big\}\cdot E_{\bar{Y}_{T_i}}\big[S_1\cdot\mathbb{1}_{H_1}\big]\Big]\\
&\le E_x\Big[\prod_{k=1}^{i-1}\mathbb{1}_{H_k}\cdot\mathbb{1}\big\{T_{i-1}<\infty\big\}\Big]\big(r+L^{27}r^8\big)\\
&\le\big(1-r^{-4}\big)^{i-1}\big(r+L^{27}r^8\big)\le 2\big(1-r^{-4}\big)^{i-1}L^{27}r^8,
\end{aligned}
$$

where the last inequality uses (3.30), and for the second-to-last inequality we used (3.27) to bound $E_{\bar{Y}_{S_i}}[T_1]$. The other expectation $E_{\bar{Y}_{T_i}}[S_1\cdot\mathbb{1}_{H_1}]$ is estimated by using Lemma 45: For $z\in B_{Lr}\backslash B_{Lr_0}$

$$
\begin{aligned}
E_z\big[S_1\cdot\mathbb{1}_{H_1}\big]&=\sum_{m=0}^{\infty}P_z\big\{S_1>m,H_1\big\}\\
&\le\sum_{m=0}^{\infty}P_z\big\{Y_k\in B_{Lr}\backslash B_{Lr_0}\text{ for }k\le m\big\}\\
&\le L^{27}r^4\sum_{j=0}^{\infty}P_z\big\{Y_k\in B_{Lr}\backslash B_{Lr_0}\text{ for }k\le jL^{27}r^4\big\}\\
&\le L^{27}r^8.
\end{aligned}
$$

We now get the bound

$$
\begin{aligned}
E_x\big[T_j,\ T_j\le U<S_j\big]&\le r\big(1-r^{-4}\big)^{j-1}+2L^{27}r^8\big(1-r^{-4}\big)^{j-2}j\\
&\le 4L^{27}r^8\big(1-r^{-4}\big)^{j-2}j,
\end{aligned}
$$

so we can calculate the first sum in (3.28) as:

$$
\begin{aligned}
\sum_{j=1}^{\infty}E_x\big[T_j,\ T_j\le U<S_j\big]&\le 4L^{27}r^8\sum_{j=1}^{\infty}\big(1-r^{-4}\big)^{j-2}j\\
&\le 4L^{27}r^8\big(1-r^{-4}\big)^{-1}r^8\\
&\le CL^{27}r^{16},
\end{aligned}
$$

for r large enough so that $r^{-4}<1/2$. Combining the above with the bound on the second sum in (3.28), we see that

$$
E_x\big[U\big]\le L^{27}r^{12}+CL^{27}r^{16}\le L^{28}r^{16}
$$

for large enough r and L. $\square$

Now, let L and r be as in Lemma 46. Define $0 = V_0 < U_1 < V_1 < U_2 < V_2 < \dots$ as the successive entrance times V_i into B_{Lr} and exit times U_i from B_{Lr} for $\bar{Y}$ if $\bar{Y}_0 = z \in B_{Lr}$. Note that $V_i = \infty$ is possible, but if $V_i < \infty$ then the walk exits in finite time by ellipticity, so $U_{i+1} < \infty$. Also, each time interval spent in B_{Lr} is of length at least 1. Then,

$$\begin{aligned}\sum_{k=0}^{n-1} P_z\{\bar{Y}_k \in B_{Lr}\} &\le \sum_{i=0}^{n} E_z\Big[(U_{i+1} - V_i)\mathbb{1}\{V_i \le n\}\Big]\\ &\le \sum_{i=0}^{n} E_z\Big[E_{\bar{Y}_{V_i}}[U_1]\mathbb{1}\{V_i \le n\}\Big]\\ &\le L^{28} r^{16} E_z\Big[\sum_{i=0}^{n} \mathbb{1}\{V_i \le n\}\Big],\end{aligned}$$

where the last inequality is by Lemma 46.

Now, we will bound the expected number of returns to B_{Lr} by the number of excursions outside B_{Lr} that fit in a time of length n

$$\begin{aligned}E_z\Big[\sum_{i=0}^{n} \mathbb{1}\{V_i \le n\}\Big] &= E_z\Big[\sum_{i=0}^{n} \mathbb{1}\Big\{\sum_{j=1}^{i}(V_j - V_{j-1}) \le n\Big\}\Big]\\ &\le E_z\Big[\sum_{i=0}^{n} \mathbb{1}\Big\{\sum_{j=1}^{i}(V_j - U_j) \le n\Big\}\Big]. \end{aligned} \tag{3.31}$$

Recall that a random vector $(\zeta_1, \dots, \zeta_n)$ stochastically dominates $(\eta_1, \dots, \eta_n)$ if

$$Ef(\zeta_1, \dots, \zeta_n) \ge Ef(\eta_1, \dots, \eta_n)$$

for any function f that is coordinatewise nondecreasing. If the process $\{\zeta_i : 1 \le i \le n\}$ is adapted to the filtration $\{\mathcal{G}_i : 1 \le i \le n\}$, and if, for some distribution function F, $P\{\zeta_i > a \,|\, \mathcal{G}_{i-1}\} \ge 1 - F(a)$, then $\{\eta_i\}$ can be taken as i.i.d. from the F-distribution.

Lemma 47 *For L and r as in Lemma 46, the excursion lengths $\{V_j - U_j : 1 \le j \le n\}$ stochastically dominate i.i.d. variables $\{\eta_j\}$ whose common distribution satisfies $P\{\eta \ge n\} \ge 1/(CL\sqrt{n})$ for $L > C$ and $L^{15} \le n \le CL^{-2}(1-\kappa)^{-r/2}$ for some constant C independent of L and r.*

Proof:

Let $V = V_1$. We know that $P_z\{V_j - U_j \ge n \,|\, \mathcal{F}_{U_j}\} = P_{\bar{Y}_{U_j}}\{V \ge n\}$, so we will bound $P_x\{V \ge n\}$ uniformly below for $x \notin B_{Lr}$. Fix $x \notin B_{Lr}$ and $1 \le j \le d$ such that $x^j \notin [-Lr, Lr]$. Assume without loss of generality that $x^j > Lr$.

Define $\bar{\bar{w}} = \inf\{n \geq 1 : \bar{\bar{Y}}_n^j \leq Lr\}$, the first time $\bar{\bar{Y}}^j$ enters the half-line $(-\infty, Lr]$. If $\bar{Y}$ and $\bar{\bar{Y}}$ start at x and stay coupled until time $\bar{\bar{w}}$, then $V \geq \bar{\bar{w}}$. As $\bar{\bar{Y}}$ is symmetric and can be translated, we can shift the origin to x^j and use results about the first entrance time $\bar{\bar{T}} = \inf\{n \geq 1 : \bar{\bar{Y}}_n^j < 0\}$. Then,

$$P_{x^j}\big\{\bar{\bar{w}} \geq n\big\} \geq P_{r+1}\big\{\bar{\bar{w}} \geq n\big\} = P_0\big\{\bar{\bar{T}} \geq n\big\}.$$

Lemma 48 *Under the same assumptions as Lemma 47,* $P_0\{\bar{\bar{T}} \geq n\} \geq \frac{1}{CL\sqrt{n}}$.

Then,

$$\begin{aligned}
P_x\big\{V \geq n\big\} &\geq P_{x,x}\big\{V \geq n,\ \bar{Y}_k = \bar{\bar{Y}}_k \text{ for } k = 1, \ldots, \bar{\bar{w}}\big\} \\
&\geq P_{x,x}\big\{\bar{\bar{w}} \geq n,\ \bar{Y}_k = \bar{\bar{Y}}_k \text{ for } k = 1, \ldots, \bar{\bar{w}}\big\} \\
&\geq P_{x^j}\big\{\bar{\bar{w}} \geq n\big\} - P_{x,x}\big\{\bar{Y}_k \neq \bar{\bar{Y}}_k \text{ for some } k \in \{1, \ldots, \bar{\bar{w}}\}\big\} \\
&\geq \frac{1}{CL\sqrt{n}} - Cb^{-Lr} \\
&\geq \frac{1}{CL\sqrt{n}}
\end{aligned}$$

if $n \leq CL^{-2}b^{2Lr}$. Since this lower bound does not depend on x, the lemma is proved. $\square$

Proof of Lemma 48:

A reflection argument as in (3.16) gives $P_0\{\bar{\bar{T}} \geq n\} \geq P_0\{\bar{\bar{Y}}_n^j = 0\}$.

In this proof, we abbreviate $S_n = \bar{\bar{Y}}_n^j/2$ and $\sigma^2 = E_0[S_1^2]$. Note that S_1 is symmetric with range spanning $\mathbb{Z}$. Define $\varphi(\theta) = E_0[e^{i\theta S_1}]$, the characteristic function of S_1. Now, write

$$\begin{aligned}
\sqrt{2\pi n}\, P_0\big\{S_n = 0\big\} &= \sqrt{\frac{n}{2\pi}} \int_{-\pi}^{\pi} \varphi^n(\theta)\, d\theta \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \varphi^n\left(\frac{\theta}{\sqrt{n}}\right) d\theta \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi}} \int_{|\theta|<A} e^{-\theta^2\sigma^2/2}\, d\theta, \\
I_2 &= \frac{1}{\sqrt{2\pi}} \int_{|\theta|<A} \left(\varphi^n\left(\frac{\theta}{\sqrt{n}}\right) - e^{-\theta^2\sigma^2/2}\right) d\theta, \\
I_3 &= \frac{1}{\sqrt{2\pi}} \int_{A<|\theta|<s\sqrt{n}} \varphi^n\left(\frac{\theta}{\sqrt{n}}\right) d\theta, \quad \text{and}
\end{aligned}$$

$$I_4 = \frac{1}{\sqrt{2\pi}} \int_{s\sqrt{n}<|\theta|<\pi\sqrt{n}} \varphi^n\left(\frac{\theta}{\sqrt{n}}\right) d\theta.$$

Fix a constant $B > 0$ and let $A = B/\sigma$. Also, let $s^2 = 1/(CL^3)$, where C is such that $1/(CL^3) \leq \sigma^2/E_0[S_1^4]$, which is possible by Property 35.

We have

$$I_1 = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} \int_{|\theta|<B} e^{-\theta^2/2}\, d\theta \geq \frac{1}{2\sigma}$$

for B large.

Now, we focus on I_3. For $|\theta| \leq s\sqrt{n}$,

$$\frac{\theta^4}{4!n^2} E_0\big[S_1^4 e^{i\zeta(\theta S_1/\sqrt{n})}\big] \leq \frac{\theta^2 s^2 E_0\big[S_1^4\big]}{4!n} \leq \frac{\theta^2\sigma^2}{4!n}$$

and $\big|\varphi^n(\theta/\sqrt{n})\big| \leq e^{-\theta^2\sigma^2/4}$. Thus,

$$\big|I_3\big| \leq \frac{1}{\sqrt{2\pi}} \int_{|\theta|>A} e^{-\theta^2\sigma^2/2}\, d\theta = \frac{\sqrt{2}}{\sigma}\frac{1}{\sqrt{2\pi}} \int_{|\theta|>B/\sqrt{2}} e^{-\theta^2}\, d\theta \leq \frac{1}{12\sigma}$$

for B large.

Next, we bound I_2. Since the walk is symmetric,

$$\varphi(\theta) = 1 - \frac{\theta^2\sigma^2}{2} + \frac{\theta^4}{4!} E_0\big[S_1^4 e^{i\zeta(\theta S_1)}\big],$$

where $\zeta(x)$ is a real-valued function with $|\zeta(x)| < |x|$. By Property 35, for $|\theta| \leq A = B/\sigma$, we have $\big|\theta^4 E_0[S_1^4]\big| \leq B^4 E_0[S_1^4]/\delta^4 \leq CB^4L^2$. Then

$$\begin{aligned}
\left|\varphi^n\left(\frac{\theta}{\sqrt{n}}\right) - e^{-\theta^2\sigma^2/2}\right| &= e^{-\theta^2\sigma^2/2}\left|\exp\left\{n\log\left(1 - \frac{\theta^2\sigma^2}{2n} + \mathcal{O}\left(\frac{B^4L^2}{n^2}\right)\right) + \frac{\theta^2\sigma^2}{2}\right\} - 1\right| \\
&= e^{-\theta^2\sigma^2/2}\left|e^{\mathcal{O}(B^4L^2/n)} - 1\right| \leq \mathcal{O}\big(B^4L^2/n\big)e^{-\theta^2\sigma^2/2},
\end{aligned}$$

where $\mathcal{O}(\cdot)$ is a universal bounded function. Hence, $|I_2| \leq \mathcal{O}(B^4L^2/n)I_1 \leq 1/(12\sigma)$ for n/L^2 large enough.

Lastly, we bound I_4. First, change variables back to find

$$\big|I_4\big| \leq \sqrt{\frac{n}{2\pi}} \int_{s<|\theta|<\pi} \big|\varphi(\theta)\big|^n\, d\theta.$$

We will bound $|\varphi(\theta)|$ away from 1. Since the walk is symmetric, $\varphi(\theta) = E_0[\cos\theta S_1]$. Choose L large enough so that $s < 2\pi$ and $1 - \cos(s/4) \geq s^2/64$. For a subset $D \subset \mathbb{R}$, denote by $|x - D|$ the distance from x to D. Then,

$$P_0\big\{\big|\theta S_1 - 2\pi\mathbb{Z}\big| \leq s/4\big\}\cos(s/4) - P_0\big\{\big|\theta S_1 - (2\mathbb{Z}+1)\pi\big| \leq s/4\big\}$$

$$
\begin{aligned}
&\quad - P_0\big\{\big|\theta S_1 - \pi\mathbb{Z}\big| > s/4\big\}\cos(s/4) \\
&\le \varphi(\theta) \\
&\le P_0\big\{\big|\theta S_1 - 2\pi\mathbb{Z}\big| \le s/4\big\} - P_0\big\{\big|\theta S_1 - (2\mathbb{Z}+1)\pi\big| \le s/4\big\}\cos(s/4) \\
&\quad + P_0\big\{\big|\theta S_1 - \pi\mathbb{Z}\big| > s/4\big\}\cos(s/4).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\big|\varphi(\theta)\big| \le{}& P_0\big\{\big|\theta S_1 - \pi\mathbb{Z}\big| > s/4\big\}\cos(s/4) \\
&+ \max\Big\{P_0\big\{\big|\theta S_1 - 2\pi\mathbb{Z}\big| \le s/4\big\} - P_0\big\{\big|\theta S_1 - (2\mathbb{Z}+1)\pi\big| \le s/4\big\}\cos(s/4), \\
&P_0\big\{\big|\theta S_1 - (2\mathbb{Z}+1)\pi\big| \le s/4\big\} - P_0\big\{\big|\theta S_1 - 2\pi\mathbb{Z}\big| \le s/4\big\}\cos(s/4)\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
1 - \big|\varphi(\theta)\big| \ge{}& P_0\big\{\big|\theta S_1 - \pi\mathbb{Z}\big| > s/4\big\}\big(1 - \cos(s/4)\big) \\
&+ \big(1+\cos(s/4)\big)\min\Big\{P_0\big\{\big|\theta S_1 - (2\mathbb{Z}+1)\pi\big| \le s/4\big\}, P_0\big\{\big|\theta S_1 - 2\pi\mathbb{Z}\big| \le s/4\big\}\Big\} \\
\ge{}& \frac{s^2}{64}P_0\big\{\big|\theta S_1 - \pi\mathbb{Z}\big| > s/4\big\} \\
&+ \min\Big\{P_0\big\{\big|\theta S_1 - (2\mathbb{Z}+1)\pi\big| \le s/4\big\}, P_0\big\{\big|\theta S_1 - 2\pi\mathbb{Z}\big| \le s/4\big\}\Big\}.
\end{aligned}
$$

If $P_0\{|\theta S_1 - \pi\mathbb{Z}| > s/4\} > 1/5$, then $1 - \big|\varphi(\theta)\big| > s^2/320$. Otherwise, $P_0\{|\theta S_1 - \pi\mathbb{Z}| > s/4\} \le 1/5$, and note that if $|\theta x - 2\pi\mathbb{Z}| \le s/4$, then there exists $z \in \mathbb{Z}$ such that $-s/4 \le \theta x - 2\pi z \le s/4$. If $\theta \ge s$, then

$$
\begin{aligned}
&\frac{s}{4} \le s - \frac{s}{4} \le \theta - \frac{s}{4} \le \theta(x+1) - 2\pi z \le \theta + \frac{s}{4} \le \pi + \frac{s}{4} < 2\pi - \frac{s}{4} \quad \text{and} \\
&-2\pi + \frac{s}{4} \le -\pi - \frac{s}{4} \le -\theta - \frac{s}{4} \le \theta(x-1) - 2\pi z \le \frac{s}{4} - \theta \le -\frac{3s}{4} \le -\frac{s}{4},
\end{aligned}
$$

so $|\theta(x+1) - 2\pi\mathbb{Z}| > s/4$ and $|\theta(x-1) - 2\pi\mathbb{Z}| > s/4$. The same holds if $\theta < -s$. Similarly, if $|\theta x - \pi(2\mathbb{Z}+1)| \le s/4$ then $|\theta(x+1) - \pi(2\mathbb{Z}+1)| > s/4$ and $|\theta(x-1) - \pi(2\mathbb{Z}+1)| > s/4$. Then,

$$
\begin{aligned}
1 - \frac{1}{5} &\le P_0\big\{\big|\theta S_1 - 2\pi\mathbb{Z}\big| \le s/4\big\} + P_0\big\{\big|\theta S_1 - \pi(2\mathbb{Z}+1)\big| \le s/4\big\} \\
&\le P_0\big\{\big|\theta(S_1+1) - 2\pi\mathbb{Z}\big| > s/4,\ \big|\theta(S_1-1) - 2\pi\mathbb{Z}\big| > s/4\big\} \\
&\quad + P_0\big\{\big|\theta S_1 - \pi(2\mathbb{Z}+1)\big| \le s/4\big\} \\
&\le P_0\big\{\big|\theta S_1 - 2\pi\mathbb{Z}\big| > s/4\big\} + P_0\big\{\big|\theta S_1 - \pi(2\mathbb{Z}+1)\big| \le s/4\big\} + \frac{1}{5}
\end{aligned}
$$

$$\leq P_0\{|\theta S_1 - \pi\mathbb{Z}| > s/4\} + 2P_0\{|\theta S_1 - \pi(2\mathbb{Z}+1)| \leq s/4\} + \frac{1}{5}$$
$$\leq \frac{2}{5} + 2P_0\{|\theta S_1 - \pi(2\mathbb{Z}+1)| \leq s/4\}, \tag{3.32}$$

To see that $P_0\{|\theta(S_1+1) - 2\pi\mathbb{Z}| > s/4,\ |\theta(S_1-1) - 2\pi\mathbb{Z}| > s/4\} \leq P_0\{|\theta S_1 - 2\pi\mathbb{Z}| > s/4\} + 1/5$ for large L, let Z_L represent the symmetric random walk that S_1 uses for its last L steps and write

$$\begin{aligned}
&P_0\{|\theta(S_1+1) - 2\pi\mathbb{Z}| > s/4,\ |\theta(S_1-1) - 2\pi\mathbb{Z}| > s/4\}\\
&= \sum_{\substack{x,y \\ |\theta(x+y+1)-2\pi\mathbb{Z}|>s/4 \\ |\theta(x+y-1)-2\pi\mathbb{Z}|>s/4}} P_0\{\tilde{X}_{\tau_1 L - L} - X_{\tau_1 L - L} = 2y\}P_0\{Z_L = x\}\\
&\leq \sum_{\substack{x>1,y \\ |\theta(x+y-1)-2\pi\mathbb{Z}|>s/4}} P_0\{\tilde{X}_{\tau_1 L - L} - X_{\tau_1 L - L} = 2y\}P_0\{Z_L = x-1\}\\
&\quad + \sum_{\substack{x\leq -1,y \\ |\theta(x+y+1)-2\pi\mathbb{Z}|>s/4}} P_0\{\tilde{X}_{\tau_1 L - L} - X_{\tau_1 L - L} = 2y\}P_0\{Z_L = x+1\}\\
&\quad + P_0\{Z_L = 0\} + P_0\{Z_L = 1\}\\
&\leq P_0\{|\theta S_1 - 2\pi\mathbb{Z}| > s/4,\ Z_L > 0\} + P_0\{|\theta S_1 - 2\pi\mathbb{Z}| > s/4,\ Z_L \leq 0\} + \frac{C}{\sqrt{L}}\\
&\leq P_0\{|\theta S_1 - 2\pi\mathbb{Z}| > s/4\} + \frac{1}{5}
\end{aligned}$$

for large L, where we used that $P_0\{Z_L = x+1\} \leq P_0\{Z_L = x\}$ and $P_0\{Z_L = -x-1\} \leq P_0\{Z_L = -x\}$ for all $x \geq 0$. P6 from page 72 of [49] was also used to get that $P_0\{Z_L = 0\} \leq C/\sqrt{L}$. Now, line (3.32) gives

$$P_0\{|\theta S_1 - \pi(2\mathbb{Z}+1)| \leq s/4\} \geq \frac{1}{5} \geq \frac{s^2}{320}$$

by our choice of s. Similarly,

$$P_0\{|\theta S_1 - 2\pi\mathbb{Z}| \leq s/4\} \geq \frac{1}{5} \geq \frac{s^2}{320}.$$

Therefore, $1 - |\varphi(\theta)| \geq \frac{s^2}{320}$. Thus,

$$|I_4| \leq \sqrt{2\pi n}\exp\left\{-\frac{ns^2}{320}\right\} \leq \sqrt{2\pi n}\exp\left\{-\frac{n}{320CL^3}\right\}.$$

Since $n \geq L^{15}$ and L is large, we have

$$\frac{n}{\log n} \geq \sqrt{n} \geq \left(\frac{2}{15}+1\right)L^7.$$

From this and Property 35, it follows that $|I_4| \le 1/(12\sigma)$.

Consequently, $P_0\{\bar{\bar{Y}}_n^j = 0\} = P_0\{S_n = 0\} \ge 1/(4\sigma\sqrt{2\pi n}) \ge 1/(CL\sqrt{n})$. $\square$

Now, we let L and r depend on n, and collect all the above estimates to show the following Proposition.

Proposition 49 *There are constants $C > 0$ and $\eta \in (0,1)$ such that for $L = \beta \log n$ and $r = n^{\varepsilon}$, with $\beta, \varepsilon > 0$*

$$\sum_{k=0}^{n-1} P_z\{\bar{Y}_k \in B_{Lr}\} = \sum_{|y|<Lr} \left[\sum_{k=0}^{n-1} P_z\{\bar{Y}_k = y\} \right] \le Cn^{1-\eta}$$

for n large and all $z \in \mathbb{S}$.

Proof:

For n large, $L(1-\kappa)^{-2r} \ge n$. Hence, we can assume that the random variables η_j in Lemma 47 satisfy $1 \le \eta_j \le n$ since this makes the conclusion of Lemma 47 even weaker. Let

$$S_0 = 0, \quad S_k = \sum_{j=1}^{k} \eta_j$$

and $K(n) = \inf\{k : S_k > n\}$ be the number of renewals up to time n, including the renewal $S_0 = 0$. These random variables are bounded, so by Wald's identity,

$$EK(n) \cdot E\eta = ES_{K(n)} \le 2n.$$

Since

$$E\eta \ge \int_{L^{15}}^{n} \frac{C}{L\sqrt{s}}\, ds \ge CL^{-1}\sqrt{n}$$

we see that

$$EK(n) \le \frac{2n}{E\eta} \le CL\sqrt{n}.$$

We will now return to line (3.31). Since the negative of the function of $(V_j - U_j)_{1\le i\le n}$ in the expectation on line (3.31) is nondecreasing, using the stochastic dominance from Lemma 47 gives an upper bound of (3.31). Combining this with the renewal bound, we see that

$$\begin{aligned} E_z\left[\sum_{i=0}^{n} \mathbb{1}\{V_i \le n\}\right] &= E_z\left[\sum_{i=0}^{n} \mathbb{1}\left\{\sum_{j=1}^{i} (V_j - U_j) \le n\right\}\right] \\ &\le E\left[\sum_{i=0}^{n} \mathbb{1}\left\{\sum_{j=1}^{i} \eta_j \le n\right\}\right] = EK(n) \le CL\sqrt{n}. \end{aligned}$$

Gathering all the bounds from this section, we observe that

$$\begin{aligned}\sum_{k=0}^{n-1} P_z\{\bar{Y}_k \in B\} &\le L^{28} n^{16\varepsilon} E_z\Big[\sum_{i=0}^{n} \mathbb{1}\{V_i \le n\}\Big] \\ &\le CL^{29} n^{1/2+16\varepsilon} = Cn^{1-\eta},\end{aligned}$$

where $0 < \eta < \frac{1}{2} - 16\varepsilon$. This requires the conditions of Lemma 46 to be met. These are

$$(\log n^{\varepsilon})^{46\log(\beta\log n)} < n^{\varepsilon} \quad \text{and} \quad \left(\frac{2d}{\kappa}\right)^{\delta\beta\log n} + \left(\frac{1-(1-\kappa)^{\delta/4}}{4}\right)^{2(\beta\log n)^{\gamma}\log\log n^{\varepsilon}} < n^{\varepsilon},$$

which are satisfied for n large, any positive ε and β, and $\delta > 0$ small enough. □

3.6 Proof of the Central Limit Theorem

We will now show that an invariance principle holds for our process.

Proof of Theorem 24:

By Lemma 30, Proposition 31, and Proposition 49, we know that for $\beta > \lambda^{-1}$:

$$\begin{aligned}\mathbb{E}_{0,0}\big[\big|X_{[0,n)} \cap \tilde{X}_{[0,n)}\big|\big] &\le Cn^{\varepsilon}\beta\log n\,\mathbb{E}_{0,0}\Big[\sum_{k=0}^{n-1}\mathbb{1}\{Y_k \in B_{Lr}\}\Big] \\ &\le Cn^{\varepsilon}\beta\log n\,\mathbb{E}_{0,0}\Big[\sum_{k=0}^{n-1}\mathbb{1}\{\bar{Y}_k \in B_{Lr}\}\Big] \\ &\le Cn^{\varepsilon} n^{1-\eta}\beta\log n.\end{aligned}$$

Taking $\varepsilon > 0$ small enough, the conditions for Proposition 29 are met, and hence (3.4) holds, so we can now apply Theorem 25. Therefore, an invariance principle holds $\mathbb{P}_\infty$-a.s. and by Theorem 16, it also holds $\mathbb{P}$-a.s. □

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